Estimation of Weibull Generalized Renewal Process Distribution Parameters through Kullback-Leibler Divergence

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1. INTRODUCTION

Repairable systems are defined as those which can be reestablished to a typical working state by some maintenance activities other than their complete substitution when encountering a failure [1]. After repair, the component can be in five states: (1) as good as new, (2) as bad as old, (3) better than old but worse than new, (4) better than new, and, (5) worse than old [2,3].

Two processes generally used to handle the states (1) and (2) are, respectively, the Renewal processes (RP) and the non-homogeneous Poisson processes (NHPP). However, in practical reliability engineering those states are often exceptions [2,4]. Generalized Renewal Process (GRP) has arisen as a significant methodology for solving these issues by demonstrating the stochastic cycles underlying repairable systems [5,6].

Considering that an operation time until the first intervention of the repairable system follows a given probability distribution, GRP assigns a rejuvenation parameter, q, to the stochastic process to extend or reduce the operation time for the next interventions, employing a virtual age function [5]. When q = 0 the GRP corresponds to a state (1) defined as a RP. Otherwise, when q = 1, a NHPP is represented [3]. Kijima [6] defines two different types of virtual age models. The type I describes that each intervention only impacts the time since the previous failure. Type II model, on the other hand, affirms that the impact is on the complete historical lifetime of the system [6].

GRP has been applied with times to failure assumed to follow a Weibull distribution [2,3,5]. The so-called Weibull-GRP involves, apart q, a scale and a shape parameters, α and β, respectively. Kaminskiy and Krivtsow have applied a Monte Carlo type solution to estimate Weibull-GRP parameters [7]. However, since the complexity of the GRP model and the Monte Carlo simulation, the computation is quite time consuming [3]. Yañez, Joglar and Modarres [3] applied a maximum likelihood (ML) estimation approach to obtain the GRP parameters.

In contrast, this research aims to evaluate an alternative method for estimation of the Weibull-GRP parameters using the Kullback-Leibler divergence of survival functions (KLS) for the same failure data analyzed by Yañez, Joglar and Modarres [3]. This method has already been applied for the Weibull distribution [8] and in this study this will be extended to the GRP. To the best of our understanding, this is the first time that KLS will be applied to this type of distribution. This measure can be understood as a relative entropy among two probability distributions [8]. Since the assessment of KL divergence for probability density functions can also present numerical convergence issues and in reliability the primary purpose is to research the survival function of a given device, the Kullback-Liebler divergence of Survival functions "KLS" is herein taken into consideration. As implemented by Liu [9], it will be measured the divergence between the modeling of a set of sample data with Weibull-GRP and an empirical survival feature. The KLS function can be minimized to obtain estimations for the Weibull-GRP three parameters q (rejuvenation parameter), β (shape), and α (scale).

2. APPROACH FOR APPLICATION OF KLS TO ESTIMATE WEIBULL-GRP DISTRIBUTION PARAMETERS

2.1 Weibull-GRP Distribution

The concept of virtual age (A_n) is introduced in generalized renovation processes. This parameter represents the calculated age of the system immediately after the n-th repair has occurred. When $A_n = y$, the

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system has a time to failure *(n+1)*-th, namely X_n+1. This time is distributed according to the cumulative distribution function (cdf) shown in the Eq. 1 [3]:

$$F(X|A_n = y) = \frac{F(X+y) - F(y)}{1 - F(y)} \tag{1}$$

F(X) represents the cdf of the Time To First Failure (TTFF) distribution of a new component or system. Considering an inter-arrival of failures that follows a Weibull distribution, the reliability function $R(\Delta x|x_i)$ is defined as [3]:

$$R(\Delta x|x_i) = exp\left[\left(\frac{q}{\alpha}x_i\right)^{\beta} - \left(\frac{\Delta x_i + qx_i}{\alpha}x_i\right)^{\beta}\right] \tag{2}$$

where,

q = repair effectiveness parameter or rejuvenation parameter;
α = scale parameter;
β = shape parameter;
x_i = real system age;
Δx_i = time between failures.

Replacing Δx_i with t-x_i $\Delta x_i = t - x_i$, this gives:

$$R(t) = \exp\left[\left(\frac{q}{\alpha} * x_i\right)^{\beta} - \left(\frac{(t-x_i)+q*x_i}{\alpha} * x_i\right)^{\beta}\right] \tag{3}$$

It is worth noting that qx_i represents the virtual age previously presented.

2.2 Kullback-Liebler Divergence of Survival Functions

Kullback-Liebler divergence can be defined as a relative entropy between two probability distributions [8]. Given two probability density functions *f* and *g* for two continuous random variables *X* and *Y*, the Kullback-Liebler divergence of *f* in relation to *g* is defined by:

$$D(f||g) = \int_{\mathbb{R}} f(x) \ln \frac{f(x)}{g(x)} dx \tag{4}$$

Since the evaluation of Kullback-Liebler divergence for probability density functions may present numerical convergence problems, the Kullback-Liebler divergence of Survival functions “KLS” is herein considered. For such an approach, the survival functions F and G are used instead of *f* and *g*. Here, the non-parametric empirical survival function, $G_n(x)$, of a random sample of size *n* is defined as below [8]:

$$G_n(x) = \sum_{i=0}^{n-1}\left(1 - \frac{i}{n}\right) I_{[X_{(i)}, X_{(i+1)}]}(x) \tag{5}$$

where,

I: the indicator function;
$(0 = X_{(0)} \leq) X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$): the ordered samples.

Similar to Yari et al. [8] when using Kullback-Liebler divergence to estimate Weibull parameters, after defining the empirical survival function, the Kullback-Liebler divergence is adjusted so that the function value remains positive during the parameter estimation process. In that way, the function value remains positive during the parameter estimation process. Hence, the adjusted Kullback-Leibler divergence (KLS) of survival functions $G_n(x)$ and F(x) is given as follows:

$$KLS(G_n||F) = \int_0^{\infty} \left(G_n(x) \ln \frac{G_n(x)}{F(x)} - G_n(x) + F(x)\right) dx \tag{6}$$

The terms where F(x) is present, in Eq. 6, are functions of Weibull-GRP parameters q, α and β, which is the basis for the parameter estimation by minimizing the KLS.

3. RESULTS AND DISCUSSION

3.1 Validation Data

To validate the application of KLS for Weibull-GRP parameter estimation it is considered the failure data from a U.S.S Halfbeak No. 3 main propulsion motor, same used by Yañez, Joglar and Modarres [3]. Tab.1 presents this dataset.

Tab. 1 - Time between failures for U.S.S. Halfbeak's example [3]

#	TBF	#	TBF
1	860	13	367
2	1608	14	2758
3	1134	15	355
4	2703	16	1084
5	645	17	855
6	95	18	280
7	1278	19	490
8	605	20	945
9	344	21	105
10	1054	22	127
11	680	22	61
12	405	24	326

3.2 Parameter Estimation Results

The parameter estimation for the Weibull-GRP is obtained by minimizing the KLS function. In the current work, the optimization method used for KLS minimization and parameter estimation was the Differential Evolution method [10]. This technique optimizes the function by a given set of candidate solutions and creating new candidate solutions by combining existing ones. The solution is the one that presents the best fitness on the optimization problem [10]. Python is the programming language applied to solve the problem.

The space search ranges applied for each parameter are:

- q: (0.0001, 1.0);
- α: (1000, 3000) hours;
- β: (1.0, 3.0).

By using the Differential Evolution optimization method for minimizing the KLS function of q, α, β the results presented in Table 2 were obtained. It can be seen that the values obtained for q and β are similar from the ones obtained by Yañez, Joglar and Modarres [3] using the ML method. However, the α parameter is not that close. For validation purposes, the KLS function value was evaluated for the two solutions obtained with the different methods. The minor the result, the better. We can conclude that the KLS method is indeed smaller when evaluated for the parameters herein obtained.

Tab. 2 – Parameter estimation based on ML and based on minimization of KLS

Method	$\hat{q}$	$\hat{\alpha}$ (hours)	$\hat{\beta}$	KLS function value
ML	0.1460	1828.00	2.026	14925.21
Kullback-Leibler	0.1951	2960.50	1.916	13071.47

4. CONCLUSIONS

The proposition of the current work is to use an alternative method to estimate Weibull-GRP parameters, by using the Kullback-Liebler divergence of survival functions. The optimization was performed using the Differential Evolution method, which optimizes the function by a given set of candidate solutions and creating new candidate solutions by combining existing ones. The solution is the one that presents the best fitness on the optimization problem. For the analyzed data, the optimization of KLS function has provided a similar solution as presented by Yañez, Joglar and Modarres [11]. The advantage of the proposed method is that when estimating the probability distribution parameters, it does not want to solve the standard "monotonic" likelihood trouble. From a practical perspective, the GRP allows the prediction of the expected number of failures in a given time and the expected time to the next failure without having to deal with the traditional approaches of "as good as new" and "as bad as old". In addition, the parameter "q" can be used as an index of the effectiveness of the repair. Moreover, the KLS methodology is very flexible because it uses a survival function instead of a density function. It can easily estimate the survival function from the observed sample data. Finally, the KLS method can be used for any distribution and is very useful in the fields of science, engineering, and medical science.

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