Maximum Likelihood Estimates for The Parameters of the q-Weibull Distribution by an Adaptive Hybrid Artificial Bee Colony Optimization Algorithm

Meng Xu, Enrique López Droguett
University of Maryland, USA

Isis Didier Lins
Federal University of Pernambuco, Brazil

ABSTRACT: The q-Weibull distribution has been applied to the reliability analysis due to its ability in modeling bathtub curves using a single set of parameters. The q-Weibull model is based on the Tsallis non-extensive entropy, which is used to describe complex systems that demonstrate long-range interaction and long-term memory. To model data with q-Weibull, its parameters must be estimated accurately. In this work, maximum likelihood estimators (MLE) are developed because they are asymptotically efficient. However, due to the intricate system of nonlinear equations derived from the log-likelihood function and the constraints over the parameters, derivative-based optimization methods may fail to converge. Since analytical expressions cannot be derived, nature-based heuristic optimization method of artificial bee colony (ABC), which does not require derivative information in the quest for the optimum, can be used to solve the maximum likelihood optimization problem. To deal with the slow convergence of ABC, this paper proposes an adaptive hybrid ABC (AHABC) algorithm which combines Nelder-Mead simplex search method with ABC for the maximum likelihood estimates of the q-Weibull parameters. The proposed algorithm is successfully applied to one example involving failure data characterized by bathtub-shaped hazard rate function, which is adequately modeled by q-Weibull distribution.

1. INTRODUCTION

The Weibull distribution, frequently used in reliability engineering, has been generalized to a q-Weibull distribution by Picoli et al. [1] in the context of non-extensive statistical mechanics. Compared to the Weibull distribution that can only describe monotonic hazard rate functions, the q-Weibull is able to model their various behaviors, including the monotonic ones: monotonically decreasing, monotonically increasing, constant, unimodal and bathtub-shaped. Assis et al. [2] confirmed that the q-Weibull is able to reproduce a bathtub curve using a single set of parameters for its three characteristic regions. Assis et al. [2] also gave the range of the parameters that should be used for each type of curve. The parameter q, also known as entropy index in statistical mechanics, was introduced by Tsallis [3]. The Tsallis' entropy generalizes the Boltzmann-Gibbs-Shannon normal entropy with this index and is given by:

$$S_q = \frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \quad (1)$$

where W is the total number of microstates of the system, p_i are the occupation probabilities and q is a real parameter that rules the degree of generalization of the theory. The standard Boltzmann-Gibbs entropy is recovered in the limit q→1.

Picoli et al. [1] firstly introduced the q-Weibull distribution which slowly interpolates the q-Exponential and the Weibull ones. The author verified that for highway length modeling neither q-Exponential nor Weibull distributions led to a satisfactory result, only the q-Weibull one gave a good adjustment. Furthermore, the q-Weibull distribution has been successfully applied to model life data in the context of reliability engineering. Costa et al. [3] used q-Weibull distribution to properly describe time-to-breakdown data of electronic devices. Sartori et al. [5] considered a q-Weibull distribution to describe the failure rate of a compression unit in a typical natural gas recovery plant, based on time-to-failure data. It is shown that the q-Weibull distribution fits better to the life data than the classic Weibull distribution, since the q-Weibull is more general and more flexible due to the additional parameter q.

To model data with q-Weibull, its parameters must be estimated accurately. The most commonly used approach to estimate q-Weibull parameters is the least squares estimation (LSE). Picoli et al. [1] used the mean square minimum method to obtain the optimal parameters. Sartori et al. [5] and Assis et al. [2] calculated q-Weibull distribution parameters through square correlation coefficient R^2 maximization. Jose and Naik [6] provided likelihood function but claimed that it is very difficult to obtain the maximum likelihood estimates of the parameters because the equations are nonlinear.

Extensive simulation studies show that the maximum likelihood estimation (MLE) method is better than the LSE method in reliability applications when data sets are typically small or moderate in size [7]. Since the distribution of maximum likelihood parameter estimates are more precise with smaller variance, in this work, we adopt the MLE method. However, the application of MLE on q-Weibull distribution presents some difficulties. The first derivative equations of the related log-likelihood function are highly nonlinear, the equations do not have analytical solutions for the parameters' estimators. In this context, a numerical approach can be alternatively adopted. In this work, we employ an artificial bee colony (ABC) algorithm, which is a nature-based heuristic method that does not require derivative information to solve the maximum likelihood problem.

ABC was introduced by Karaboga [8] and is an optimization algorithm based on the intelligent foraging behavior of honey bee swarm for optimizing multidimensional and multimodal numerical functions. However, the convergence performance of ABC for local search is slow due to its solution search equation, which is good at exploration but poor at exploitation. Some modified versions of ABC have been proposed by researchers to improve its local search performance. To mention a few, inspired by PSO, Zhu and Kwong [9] proposed an improved ABC algorithm named gbest-guided ABC (GABC) algorithm by incorporating the information of global best solution into the solution search equation to improve exploitation. Kang et al. [10] proposed a Hooke-Jeeves ABC (HABC) algorithm which combines Hooke-Jeeves pattern search with ABC algorithm. In the HABC, the exploration phase is performed by ABC and the exploitation phase is completed by pattern search. Karaboga and Gorkemli [11] proposed the Quick ABC (qABC), which models the behavior of onlooker bees more accurately and improves the performance of standard ABC in terms of local search ability. Kang et al. [12] proposed a hybrid simplex ABC algorithm (HSABCA) that combines Nelder-Mead simplex method with artificial bee colony algorithm for inverse analysis problems. The HSABCA was applied to parameter identification of concrete dam-foundation systems. The Nelder-Mead simplex algorithm proposed by Nelder and Mead [13] is an efficient local search method. It was also combined with other heuristic search method to improve the convergence accuracy and speed. For example, Fan and Zahara [14] proposed the hybrid NM-PSO algorithm based on the Nelder-Mead simplex search method and particle swarm optimization for unconstrained optimization.

A method that not only does not depend on derivative but also presents fast convergence is necessary in the MLE optimization problem. In this direction, this paper proposes an Adaptive Hybrid ABC (AHABC) algorithm which combines a local Nelder-Mead simplex search method with ABC to enhance the local search capability of ABC. AHABC dynamically controls the exploration and exploitation, given that the parameter for Nelder-Mead local search is adaptively tuned according to the search status.

This paper is organized as follows. In Section 2, an introduction about the q-Weibull distribution and its properties are given. In Section 3, the maximum likelihood problem related to the q-Weibull distribution is presented. In Section 4, AHABC algorithm is proposed to solve the maximum likelihood estimation problem. In Section 5, the proposed AHABC is applied to one example involving reliability-related data that can be properly modeled by the q-Weibull distribution. Finally, conclusions are given in Section 6.

2. THE Q-WEIBULL DISTRIBUTION

The probability density function (PDF) of the q-Weibull distribution is as follows:

$$f_q(t) = (2-q)\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} exp_q\left[-\left(\frac{t}{\eta}\right)^{\beta}\right] \; t \geq 0 \tag{2}$$

where $\beta > 0$ and $q < 2$ are shape parameters and $\eta > 0$ is a scale parameter. The q-Exponential function is defined as:

$$\exp_q(x) = \begin{cases} (1+(1-q)x)^{\frac{1}{1-q}}, & if\ 1-(1-q)x > 0 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Therefore, the q-Weibull PDF can be rewritten as:

$$f_q(t) = (2-q)\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}\left[1-(1-q)\left(\frac{t}{\eta}\right)^{\beta}\right]^{\frac{1}{1-q}}, t \geq 0 \tag{4}$$

in which

$$t \in \begin{cases} [0,\infty), & \text{for } 1 < q < 2 \\ [0, t_{max}], & \text{for } q < 1 \end{cases} \tag{5}$$

with $t_{max} = \frac{\eta}{(1-q)^{1/\beta}}$

The q-Weibull cumulative distribution and reliability functions are as follows:

$$F_q(t) = 1-\left[1-(1-q)\left(\frac{t}{\eta}\right)^{\beta}\right]^{\frac{2-q}{1-q}} \tag{6}$$

$$R_q(t) = \left[1-(1-q)\left(\frac{t}{\eta}\right)^{\beta}\right]^{\frac{2-q}{1-q}} \tag{7}$$

The hazard function is defined as:

$$h_q(t) = \frac{f_q(t)}{R_q(t)} = \frac{(2-q)\frac{\beta}{\eta^{\beta}}t^{\beta-1}}{1-(1-q)\left(\frac{t}{\eta}\right)^{\beta}} \tag{8}$$

This equation is able to represent different types of hazard functions, according to the values of the shape parameters [2], besides the constant type (with $q \to 1$ and $\beta = 1$). The function $h_q(t)$ is monotonically decreasing for $1 < q < 2$ and $0 < \beta < 1$, monotinically increasing for $q < 1$ and $\beta > 1$, unimodal for $1 < q < 2$ and $\beta > 1$ and bathtube-shaped for $q < 1$ and $0 < \beta < 1$.

For numerical experiments, we use the inverse transform method by inverting $F_q(t)$ to generate random samples. The q-Weibull random number generator is then obtained:

$$t = \eta \cdot \left\{ \frac{\left[1 - U^{\frac{1-q}{2-q}}\right]}{1-q} \right\}^{\frac{1}{\beta}} \tag{9}$$

where U is a uniform random number in [0,1].

3. THE MAXIMUM LIKELIHOOD CONSTRAINED PROBLEM FOR THE Q-WEIBULL DISTRIBUTION

In order to estimate the parameters of the q-Weibull distribution, the MLE method is adopted. Let $t = (t_1, t_2, \dots, t_n)$ be an n-dimensional vector of observed failure times $t_i, i = 1, \dots, n$, independently drawn from a q-Weibull distribution. The likelihood function is given by:

$$L(\eta, \beta, q|t) = \prod_{i=1}^{n} f_q(t_i) = \prod_{i=1}^{n} (2-q) \frac{\beta}{\eta} \left(\frac{t_i}{\eta}\right)^{\beta-1} \left[1 - (1-q)\left(\frac{t_i}{\eta}\right)^{\beta}\right]^{\frac{1}{1-q}} \tag{10}$$

Instead of maximizing Eq. (10), it is easier to optimize its log-likelihood function as Eq. (11). The optimization problem is constrained to the conditions that guarantee the properties of the q-Weibull PDF as Eq. (12)-(15). The maximum likelihood constrained optimization problem for the q-Weibull distribution is given as follows:

max

$$\mathcal{L}(\eta, \beta, q|t) = nln(2-q) + nln(\beta) - n\beta \ln(\eta) + (\beta - 1) \sum_{i=1}^{n} \ln(t_i) + \frac{1}{1-q} \sum_{i=1}^{n} \ln[1 - (1-q)\left(\frac{t_i}{\eta}\right)^{\beta}] \tag{11}$$

$$\text{s.t. } 2 - q > 0 \tag{12}$$

$$1 - (1-q)\left(\frac{t_i}{\eta}\right)^{\beta} > 0, i = 1, \dots, n \tag{13}$$

$$\eta > 0 \tag{14}$$

$$\beta > 0 \tag{15}$$

The first derivatives of log-likelihood function for parameters are nonlinear, and we cannot obtain analytical solutions. A heuristic based constrained optimization method can solve this problem. In this paper, the maximum likelihood estimates $\hat{\eta}, \hat{\beta}$ and $\hat{q}$ are obtained by means of AHABC algorithm, which is described in the following section.

4. PROPOSED ADAPTIVE HYBRID ARTIFICIAL BEE COLONY ALGORITHM

4.1 Overview of Artificial Bee Colony Algorithm

In ABC, the colony consists of three groups of bees: employed bees, onlookers and scouts. The position of a food source represents a possible solution to the optimization problem and the nectar amount of a food source corresponds to the fitness of the associated solution. At the beginning, the algorithm generates a randomly distributed initial population of SN solutions. Each solution x_i ($i = 1,2, \dots, SN$) is an n-dimensional vector.

A candidate solution v_i from the old one x_i can be generated as:

$$v_{ij} = x_{ij} + \phi_{ij}(x_{ij} - x_{kj}) \tag{16}$$

where $k \in \{1,2, \dots, SN\}$ and $j \in \{1,2, \dots, D\}$ are randomly chosen indexes; k is different from i; ϕ_{ij} is a random number in the range [-1,1].

The fitness of a solution $fit(x_i)$ can be calculated from its objective function value $f(x_i)$ as:

$$fit(x_i) = \begin{cases} \frac{1}{1+f(x_i)}, & \text{if } f(x_i) \geq 0 \\ 1 + \text{abs}\big(f(x_i)\big), & \text{if } f(x_i) < 0 \end{cases} \tag{17}$$

An onlooker bee chooses a solution depending on the probability value p_i associated with food source i as follows:

$$p_i = \frac{fit(x_i)}{\sum_{n=1}^{SN} fit(x_n)} \tag{18}$$

After a candidate solution is produced and the fitness is evaluated, its performance is compared with that of its old one. If the fitness value of the new solution v_i is higher than the current solution x_i, it replaces the current solution, otherwise the current solution remains. When a solution cannot be improved further through a predetermined number of cycles, called '$limit$', then that solution is abandoned and replaced with a new solution generated randomly by a scout as:

$$x_{i,j} = x_{min,j} + rand(0,1)(x_{max,j} - x_{min,j}) \tag{19}$$

where $x_{min,j}$ and $x_{max,j}$ are lower and upper bounds for j^{th} dimension.

4.2 Overview of Nelder-Mead Simplex Algorithm

The Nelder-Mead simplex algorithm was developed by Nelder and Mead [13] to efficiently find local minima. This algorithm uses a simplex of $D + 1$ points for D dimensional vectors. The main idea is to collaboratively move $D + 1$ vertices to the lowest point of the objective function. This method rescale the simplex by four procedures: reflection, expansion, contraction and shrinkage. Let $x_1, x_2, \dots, x_{D+1}$ represent the points in one simplex, ranking from the best one to the worst one.

One iteration of simplex search can be described as either of two steps:

1) Replace the worst point

The candidates to replace the worst point x_{D+1} in previous simplex for the next iteration, and these candidates are:

$$x_r = x_o + \alpha(x_o - x_{D+1}) \tag{20}$$

$$x_e = x_o + \gamma(x_o - x_{D+1}) \tag{21}$$

$$x_c = x_o + \rho(x_o - x_{D+1}) \tag{22}$$

where, $\alpha = 1, \gamma = 2, \rho = -0.5$, which are respectively reflection, expansion and contraction coefficients [12]; x_0 is the center of points $\{x_1, x_2, \dots, x_D\}$. The new $(D+1)^{th}$ point is the best candidate, i.e., $x_{D+1} \in \{x_r, x_e, x_c\}$ and $f(x_{D+1}) = \min\{f(x_r), f(x_e), f(x_c)\}$.

2) Shrink the simplex towards the best point x_1

All points except x_1 will be reduced towards x_1, i.e.

$$x_i = x_1 + \delta * (x_i - x_1), i = 2,3, \dots, D+1 \tag{23}$$

where $\delta = 0.5$ is the shrink coefficient [12].

From these two steps, we can see that the simplex search is just exploiting local neighborhood area and is very aggressive towards the local minimum.

4.3 The Adaptive Hybrid ABC for the q-Weibull MLE problem

In the ABC algorithm, while onlookers and employed bees carry out the exploitation process in the search space, the scouts control the exploration process. From our simulation experiments, we found that although ABC could find the global optimum very fast, the convergence speed of ABC for local search is slow. In order to make full use of ABC's exploration, and avoid its drawbacks, an adaptive hybrid ABC is proposed, which incorporates local search stage. The main idea of AHABC is that through adaptively tuning the parameters of hybrid ABC according to the search process, the hybrid ABC will gradually change from global ABC search pattern to local search pattern. The details of this AHABC algorithm are presented in the following subsections.

4.3.1 Hybrid Strategy

"Hybrid Strategy" is the method to combine ABC with a local search algorithm. In our proposed algorithm, the Nelder-Mead simplex local search is incorporated into ABC as an additional phase after the original three phases within every iteration. The input of local search phase is the best $D + 1$ solutions in the population, where D is dimension of the optimization problem. These solutions will be exploited by the Nelder-Mead simplex local search for a number of function evaluations NS.

4.3.2 Adaptive Switch Mechanism

"Adaptive switch mechanism" describes the mechanism how the hybrid algorithm is changing from global exploration to local exploitation. Basically, the principle of "adaptive switch mechanism" is to gradually increase the use of local search by tuning algorithm parameters according to the search process. In this paper, we propose the following formula to determine the number of function evaluations for simplex search:

$$NS = C * limit * \text{total number of scout bees.} \tag{24}$$

Firstly, this definition of NS will guarantee that the search process will become more and more local. Secondly, the total number of scout bees is a symbol of search status. A large number of scout bees indicates that a significant portion of the solution space has been explored, that the exploration is becoming inefficient and a local exploitation is becoming urgent. The $limit$ is also an important ABC parameter, which controls the scout bee generation frequency. C is a coefficient that controls the amount of local search. For the optimization problem in this work, $C = 1$ provided an acceptable convergence speed. Thus, we use the product of $limit$ and the total number of scout bees as the number of function evaluations within the local search phase of the AHABC. In summary, NS dynamically increases along the search process and it gradually changes from global to local.

4.3.3 Constraints

For the constraints (12-15) related to the q-Weibull MLE problem, we adopt the “throw away” approach, which means that if the generated solution is not feasible, we throw it away and keep the current solution. Basically, this is a simplified Deb’s rule [15] that involves domination rules between solutions.

There are three commonly used control parameters in the standard ABC: the number of food sources SN, the value of $limit$, which can be obtained from the formula $limit = SN * D$ [8], where D is dimension of the optimization problem, and the maximum cycle number (MCN). In the AHABC algorithm, one iteration cycle incorporates the Nelder-Mead local search iterations. Instead of setting the iteration numbers for ABC and Nelder-Mead local search separately, we use one parameter of maximum number of function evaluations (MFE), totaling the number of the ABC and Nelder-Mead local search function evaluations. There are three stop criteria:

1) Maximum number of function evaluations (MFE).
2) The global best solution is the same for $maxBestTrial$ times. In this case, the iteration number in which the best solution has been found is used.
3) The global best objective function value in two consecutive iterations are different, but such a difference is less than a predefined tolerance ε.

5. APPLICATION EXAMPLES

In this section, the proposed AHABC for the maximum likelihood estimates of the q-Weibulll parameters is applied to one example involving reliability-related data of engineering equipment. For this application example, AHABC parameters are shown in Table 1. Also, the initial bounds for parameters q, β and η are set to [-10, 1.9], [0.1, 10], [0.1, t_{mean}], respectively, where t_{mean} is the mean of sample.

Table 1-AHABC parameters

Parameter		Value
ABC	SN	50
	$limit$	150
	MFE	200,000
	$maxBestTrial$	1000
	ε	1e-16
Nelder-Mead simplex method	α	1
	γ	2
	ρ	-0.5
	δ	0.5
Adaptive hybrid coefficient	C	1

Failure data of oil well pump [2] in Table 2 is analyzed. The objective is to obtain the maximum likelihood estimates for the q-Weibull parameters by means of the proposed AHABC.

Table 2-Times to failure of pumps

8	38	42	59	71	146	184
185	199	204	214	379	457	457
494	515	568	680	684	808	964

The AHABC is replicated 30 times. The estimated MLE parameters and the associated standard deviations are shown in Table 3. For these estimates, the PDF, the reliability function and the bathtub-shaped hazard rate function are presented in Figure 1-Figure 3.

Table 3-maximum likelihood estimates for 30 replication of AHABC

	Mean	Std
$\hat{q}$	-2.1910	4.5853E-07
$\hat{\beta}$	0.7726	1.6977E-08
$\hat{\eta}$	4455.2019	9.2597E-04
L	-142.2998	9.6310E-14

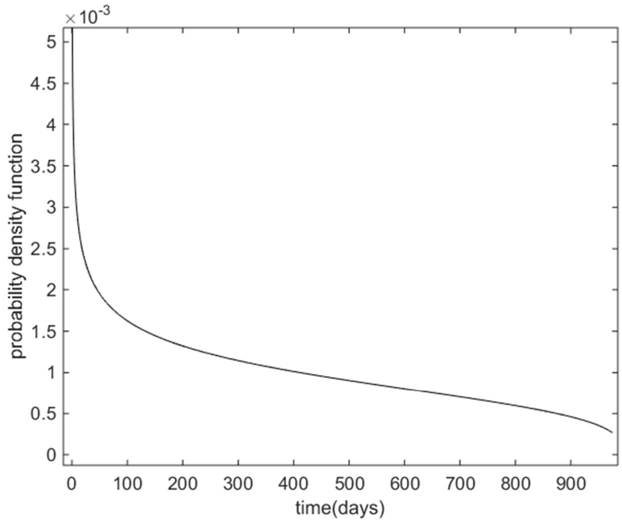

Figure 1 – q-Weibull probability density function

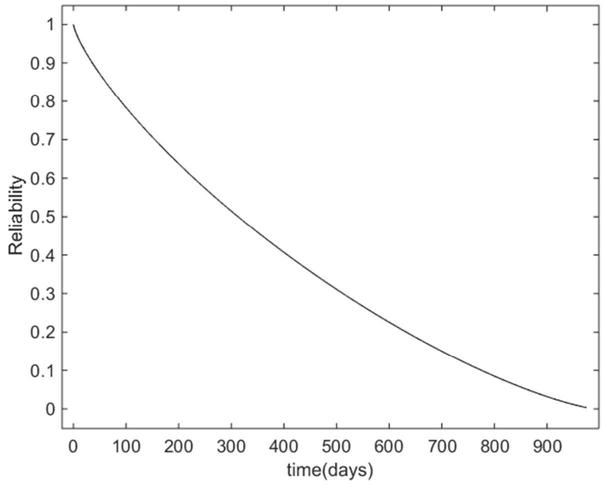

Figure 2 – q-Weibull reliability function

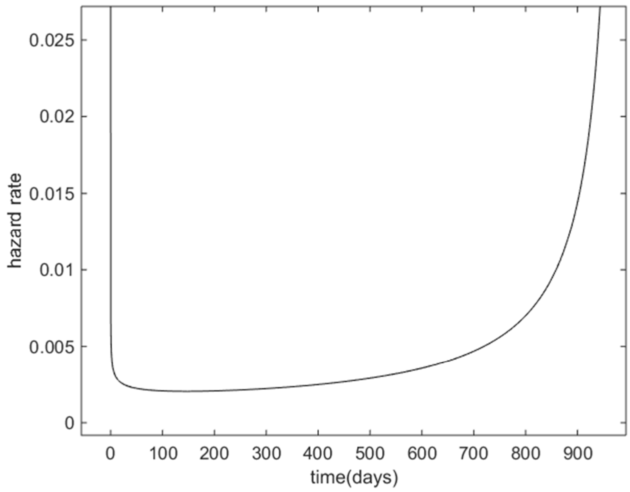

Figure 3 – q-Weibull hazard rate function

To test the goodness-of-fit, we use the Kolmogorov-Smirnov (KS) test, which compares the empirical and the cumulative distribution function (CDF). However, the traditional KS test is not applicable in our situation where the parameters of the theoretical distribution have been estimated from the same bunch of data used to apply this goodness-of-fit test [16]. Therefore, a bootstrapped version of the KS test [17] has been developed and applied. The KS test statistic is computed as follows:

$$D^0 = max\left|\left|F_n(t_i) - F(t_i|\hat{q},\hat{\beta},\hat{\eta}))\right|,\left|F_n(t_{i-1}) - F(t_i|\hat{q},\hat{\beta},\hat{\eta})\right|\right| \tag{25}$$

where $F_n(t_i) = i/n$ is the empirical CDF and $F(t_0) = 0$, $F(t_i|\hat{q},\hat{\beta},\hat{\eta})$ is theoretical CDF with estimated parameters. B bootstrap samples $t^j = \{t_1^j, t_2^j, \ldots, t_n^j\}, j = 1,2,\ldots,B$ are generated using Eq. (9) with $\hat{q},\hat{\beta},\hat{\eta}$. The maximum likelihood estimates $\hat{q}^j,\hat{\beta}^j,\hat{\eta}^j$ for the j^{th} sample are obtained by the proposed

AHABC. The test statistic D^j is computed with $F(t_i^j|\hat{q}^j,\hat{\beta}^j,\hat{\eta}^j)$ in place of $F(t_i|\hat{q},\hat{\beta},\hat{\eta})$. Then, we get $B+1$ observations of the KS test statistic D. The p-value is computed as the number of observations where D^j exceeds D^0 divided by $B+1$.

In this example, $B = 999$, $n = 21$, $D^0 = 0.1431$ and $p = 0.4160$. With such large p-value, we cannot reject the hypothesis that data from Table 2 follows the estimated q-Weibull distribution. Figure 4 presents the empirical and estimated CDFs of the original sample data.

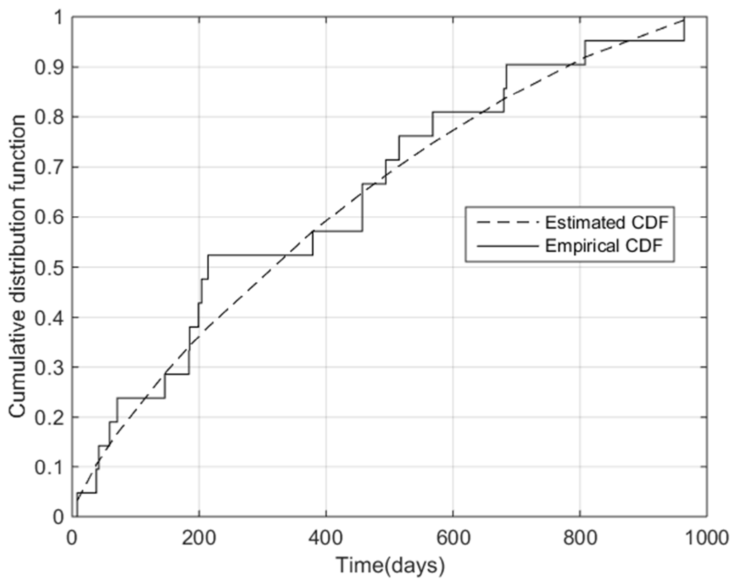

Figure 4 – Empirical and estimated CDFs

6. CONCLUSIONS

This paper presents a novel numerical optimization algorithm to obtain the maximum likelihood estimates of q-Weibull parameters, which cannot be analytically solved. An adaptive hybrid artificial bee colony (AHABC) algorithm is proposed, which combines the global exploration of ABC and the local exploitation of Nelder-Mead simplex search. More importantly, in order to dynamically control the exploration and exploitation, the number of function evaluations for local search in one ABC iteration is adaptively tuned according to the search status, indexed by the product of total number of scout bees and limit value. The proposed algorithm is applied to an example involving failure data characterized by bathtub-shaped hazard rate functions, which are adequately modeled by the q-Weibull distribution.

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