# q-Weibull-based generalized renewal process

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## 1. INTRODUCTION

A repairable system, characterized as being restorable with no need of complete replacement, can reach five states following a repair: as good as new, as bad as old, better than old but worse than new, better than new and worse than old [1]. Traditional probabilistic models in literature of repairable system analysis account for the states as good as new and as bad as old condition, which are modeled by renewal process (RP) and non-homogeneous Poisson process (NHPP), respectively. Nevertheless, these states are often exceptions rather than rule, from the standpoint of practical reliability engineering [2].

In this context, Kijima & Sumita [3] have proposed a probabilistic model, named generalized renewal process (GRP), which is able to attend all the post-repair states due to the inclusion of the parameter of repair effectiveness. This parameter, denoted in this paper as $r$, represents the post-repair states through the notion of virtual age [4]. GRP has been widely applied using times to failure assumed to be Weibull random variables [1,2,5,6].

Although the Weibull distribution has been widely used along with GRP, the q-Weibull probability distribution appears as an interesting alternative to be used in the GRP. The q-Weibull is proposed as a distribution which smoothly interpolates the q-Exponential and the Weibull in order to generate a unified framework to accommodate different cases of data adjustment [7]. The Weibull distribution can handle monotonically decreasing, constant and monotonically increasing hazard rate functions, whereas, besides these three behaviors, the q-Weibull distribution can model two additional ones with a single set of parameters: unimodal and U-shaped (bathtub curve) [8]. Therefore, it is expected that the q-Weibull into GRP would bring more flexibility. This flexibility is due to the $q$ parameter, which controls the shape of the distribution along with the $\beta$ parameter, while the Weibull distribution has just $\beta$ affecting its shape.

In the proposed GRP model type I, the time to the occurrence of the first failure are distributed according to a q-Weibull, while the subsequent failures follow a conditional q-Weibull distribution, meaning that the arrival of a subsequent failure is conditional on the cumulative operating time up to the last failure. The maximum likelihood estimates are obtained for failure-terminated cases. The estimation of GRP parameters based on the maximum likelihood method results in an intricate system of first derivatives and the parameter estimators are very difficult to be analytically obtained. Therefore the chosen estimation is to maximize the log-likelihood function by means of a Particle Swarm Optimization (PSO) algorithm. PSO is a derivative-free probabilistic heuristic based on the social behavior of biological organisms which has as a basic element a particle that can fly throughout the search space of the problem toward an optimum using its own information and the information provided by other particles within its neighborhood [9].

## 2. OBJECTIVES

The main objective of this paper is to propose a GRP model based on the q-Weibull distribution. In order to estimate the model parameters, the associated log-likelihood function is optimized by PSO. The q-Weibull-based GRP is applied to failure data sets extracted from Yañez *et al.* [1] and Wang & Yang [2], which concerns a compressor and an NC machine tool, respectively.

## 3. DESCRIPTION

### *3.1 q-Weibull distribution*

The proposals of generalization of Weibull distribution often share the base on the exponential framework. Exponentials are commonly used in non-interacting or weakly interacting systems, while power-laws are used in the statistical distributions of complex systems, for instance systems that exhibit long-range (spatial) interactions, long-term (temporal) memory, among others [8].

However, component's failure usually have multiple and interacting causes, therefore a complex behavior can possibly appear. Power-law like expressions are expected to substitute exponentials in the statistical description for these cases [8]. The q-distributions optimize generalized entropies such as Tsallis nonextensive entropy, a generalization of Boltzmann-Gibbs-Shannon (BGS) entropy, introducing the possibility to extend statistical mechanics to complex systems in a coherent and natural way [7]. The q-Exponential function can be defined as:

$$exp_q(x) = \begin{cases} (1+(1-q)x)^{\frac{1}{(1-q)}}, & \text{if } (1+(1-q)x) \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

In this context, the q-Weibull distribution can be seen as natural step forward to Weibull distribution on the light of nonextensive statistics, as it is derived from the substitution of the Exponential function by a q-Exponential in the classic Weibull model, represented in Equation (2) [8]:

$$f_q(t) = (2-q)\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} exp_q\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right], \tag{2}$$

in which parameters $\beta$ and $q$, control the shape of the distribution, whereas α is the scale parameter. Differently from the classic Weibull, the q-Weibull has two parameters which affect its shape.

The support of Equation (2) changes depending on the value of $q$:

$$t \in \begin{cases} [0, \infty), & \text{for } 1 < q < 2, \\ [0, t_{max}], & \text{for } q < 1, \end{cases} \tag{3}$$

where $t_{max} = \alpha/(1-q)^{1/\beta}$ is the maximum allowed time so as to preserve the probabilistic properties of Equation (2) when $q < 1$. For these values the integration of $f_q(t)$ diverges for $t > t_{max}$ [8].

The q-Weibull distribution has other probability distributions as special cases: when $\beta = 1$, a q-Exponential distribution; for $q \to 1$, a Weibull distribution; for both $\beta = 1$ and $q \to 1$, an Exponential distribution. The q-Weibull cumulative distribution and reliability function are given by Equations (4) and (5), respectively:

$$F_q(t) = 1 - \left[exp_q\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right]^{2-q} = 1 - \left[1-(1-q)\left(\frac{t}{\alpha}\right)^{\beta}\right]^{\frac{2-q}{1-q}}, \tag{4}$$

$$R_q(t) = \left[exp_q\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right]^{2-q} = \left[1-(1-q)\left(\frac{t}{\alpha}\right)^{\beta}\right]^{\frac{2-q}{1-q}} \tag{5}$$

Assis *et al.* [8] list the combination of $\beta$ and $q$ values representing the various types of hazard rate function behaviors that can be reproduced by the q-Weibull distribution (Table 1): monotonically decreasing, constant, monotonically increasing, unimodal and U-shaped (bathtub curve). The hazard rate function is given by Equation (6):

$$h_q = \frac{f_q}{R_t} = (2-q)\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}\left[exp_q\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right]^{q-1} = (2-q)\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}\left[1-(1-q)\left(\frac{t}{\alpha}\right)^{\beta}\right]^{-1}. \quad (6)$$

Table 1 – Behaviors of hazard rate in the q-Weibull distribution [5]

| | $\mathbf{0 < \beta < 1}$ | $\boldsymbol{\beta} = \mathbf{1}$ | $\boldsymbol{\beta} > \mathbf{1}$ |
|---|---|---|---|
| $\boldsymbol{q} < \mathbf{1}$ | Bathtub curve | Monotonically increasing | Monotonically increasing |
| $\boldsymbol{q} = \mathbf{1}$ | Monotonically decreasing | Constant | Monotonically increasing |
| $\mathbf{1} < \boldsymbol{q} < \mathbf{2}$ | Monotonically decreasing | Monotonically decreasing | Unimodal |

### *3.2 Generalized Renewal Process*

The GRP, probabilistic model proposed by Kijima & Sumita [3], is able to incorporate the five post-repair states that a repairable system may assume. The models that have been mostly used in the reliability analysis of repairable systems are the RP and the NHPP, which can be considered particular cases of GRP [1]. However, RP and NHPP assume simplifying hypotheses which restrict its application to realistic cases.

The RP assumes that the failures are independent and identically distributed, therefore the system returns to an as good as new condition, representing an ideal situation [1]. This situation may only occur when the system is completely replaced after the failure, resembling to non-repairable systems.

In the NHPP the time between failures follows a conditional exponential probability function, meaning that the arrival of the $i$th failure is conditional on the cumulative operation time up to the $i-1$ failure. It is assumed that the system condition is as bad as old after a repair [1].

By covering major repair assumptions encountered in practice, GRP provides more flexibility in modeling real life failure occurrence processes [5]. Figure 1 presents a categorization of stochastic point processes for modeling repairable systems.

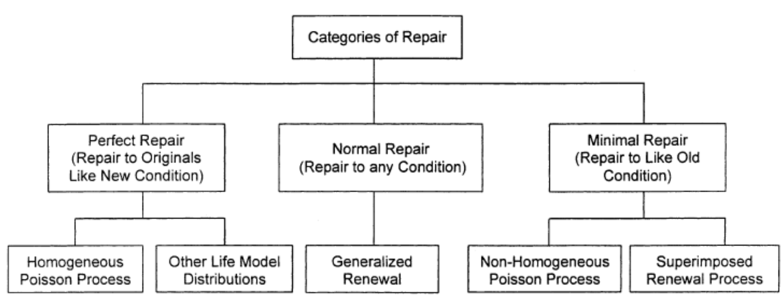


Figure 1 – Categories of stochastic point processes for repairable systems [1]

GRP presents the concept of virtual age, which is illustrated in Figure 2. If $x_i$ and $y_i$, in Figure 2, are the equipment's calculated age before and after repair, respectively, and $t_i$ is the chronological time, it is possible to verify the relation between the virtual age of the system and the real age, according to parameter $r$, defined as the repair effectiveness [6].

The values of the parameter $r$ can be seen as an index for representing effectiveness and quality of repair. Assuming $r = 0$ leads to an RP (as good as new condition), while $q = 1$ leads to an NHPP (as bad as old condition). The intermediate values $0 < r < 1$ lead to a condition of better than old but worse than new. When $r > 1$, the equipment's condition is worse than old; and when $r < 0$, the equipment's associated condition is better than new [1,6].

Accordingly to Kijima [10], two models can be constructed depending on how the repair activities affect the virtual age process. In the first model, it is assumed that the $n$th repair cannot remove the damages incurred before the $(n-1)$th repair. The virtual age type I is given by Equation (7). In the second model, virtual age type II, at the $n$th failure the virtual age has been accumulated to $A_{n-1} + X_n$, as defined in Equation (8):

$$A_n = A_{n-1} + r\, X_n, \tag{7}$$

$$A_n = r\,(A_{n-1} + X_n). \tag{8}$$

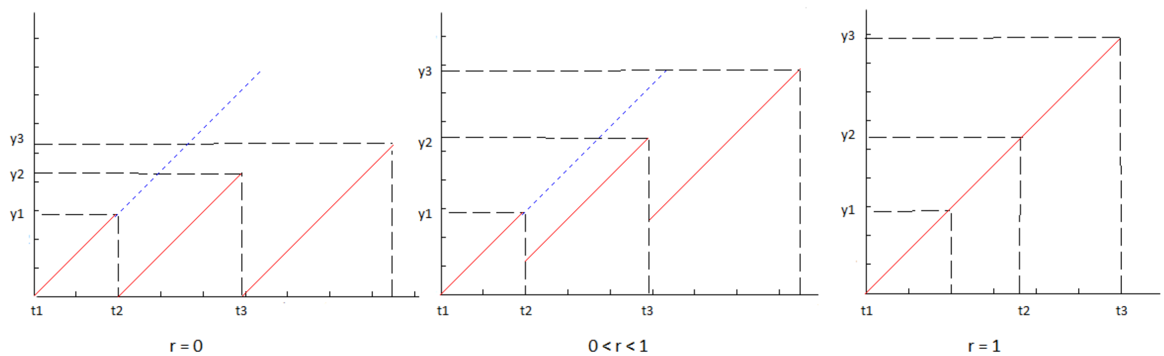


Figure 2 – Virtual age and the repair index

Kijima type I assumes that the $n$th repair can only compensate for the damage accumulated during the period of time between the $n$th and $(n-1)$th failure, while Kijima type II assumes that the repair can compensate the system damage since the beginning of its operation [2]. It is recommended for complex systems to be modelled using Kijima type II GRP model, while individual components should be modelled using Kijima type I GRP model [4]. In this paper an individual component is analyzed, thus the virtual age type I is applied.

### *3.3 Maximum Likelihood Estimators for q-Weibull-GRP parameters*

In order to obtain the maximum likelihood estimators for the proposed q-Weibull GRP model, the definition of conditional probability is used:

$$P(T \le t | T > t_1) = \frac{F(t) - F\,(t_1)}{R(t_1)} = \frac{1 - R(t) - 1 + R(t_1)}{R(t_1)} = 1 - \frac{R(t)}{R(t_1)}. \tag{9}$$

Where $F(\cdot)$ and $R(\cdot)$ are, respectively, the probability distribution of component failure and reliability at the respective times. Assuming a q-Weibull distribution Equation (9) turns into:

$$F(t_i | t_{i-1}) = 1 - \frac{\left[exp_q\left(-\frac{t_i}{\alpha}\right)^{\beta}\right]^{2-q}}{\left[exp_q\left(-\frac{t_{i-1}}{\alpha}\right)^{\beta}\right]^{2-q}}. \tag{10}$$

The conditional q-Weibull density function is:

$$f(t_i|t_{i-1}) = (2-q)\ \frac{\beta}{\alpha}\ \left(\frac{t_i}{\alpha}\right)^{\beta-1}\ exp_q\left(-\frac{t_i}{\alpha}\right)^{\beta}\ \left[exp_q\left(-\frac{t_{i-1}}{\alpha}\right)^{\beta}\right]^{q-2}. \tag{11}$$

When the Kijima's virtual age type I is introduced to Equation (11) it becomes:

$$f(t_i|t_{i-1}) = (2-q)\ \frac{\beta}{\alpha}\ \left(\frac{t_i + r\ \sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta-1}\ exp_q\left(\frac{-t_i - r\ \sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\left[exp_q\left(-\frac{r\ \sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^{q-2}. \tag{12}$$

Equations (9)-(11) are valid for the subsequent $i-1$ observations after the first failure occurrence.

### 3.3.1 Failure-terminated GRP maximum likelihood estimators

Failure-terminated cases are the occasions when failure data are available up to the time at the last failure occurrence. Considering that the first failure doesn't attend to the conditional probability function, then, the likelihood function is given by:

$$L = f(t_1)\ \prod_{i=2}^{n} f(t_i). \tag{13}$$

By substituting Equations (2) and (12) in Equation (13), the maximum likelihood function can be written as

$$L(\alpha,\beta,q,r|t) = (2-q)\ \frac{\beta}{\alpha}\ \left(\frac{t_1}{\alpha}\right)^{\beta-1}\ \left[1-(1-q)\left(\frac{t_1}{\alpha}\right)^{\beta}\right]^{\frac{1}{(1-q)}} \prod_{i=2}^{n}(2-q)\ \frac{\beta}{\alpha}\ \left(\frac{t_i + r\ \sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta-1}\ \left[1-(1-q)\left(\frac{t_i + r\ \sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^{\frac{1}{1-q}}\ \left[1-(1-q)\left(\frac{r\ \sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^{\frac{q-2}{1-q}}. \tag{14}$$

and the corresponding log-likelihood function actually used as objective function is:

$$\mathcal{L}(\alpha,\beta,q,r|t) = (\beta-1)\ln t_1 + (1-\beta-n)\ln\alpha + n\,(\ln\beta + \ln(2-q)) + \frac{1}{(1-q)}\left[1-(1-q)\left(\frac{t_1}{\alpha}\right)^{\beta}\right] + \sum_{i=2}^{n}\left\{\frac{(q-2)}{(1-q)}\ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] + (\beta-1)\ln\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right) + \frac{1}{(1-q)}\ln\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]\right\}. \tag{15}$$

The procedure adopted to find the maximum likelihood estimators would be to differentiate the log-likelihood function with respect to each of the parameters, make the derivatives equal to zero and solve the resulting system of equations. However, the resulting system involves intricate nonlinear equations and analytical expressions for the estimators cannot be obtained. Therefore, in this paper, a constrained optimization method based on PSO heuristic is adopted. The optimization problem must be solved to find the maximum likelihood estimators is:

$$\max_{\alpha,\beta,q,r} (\beta-1)\ln t_1 + (1-\beta-n)\ln\alpha + n\left(\ln\beta + \ln(2-q)\right) + \frac{1}{(1-q)}\left[1-(1-q)\left(\frac{t_1}{\alpha}\right)^{\beta}\right] + \sum_{i=2}^{n}\left\{\frac{(q-2)}{(1-q)}\ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right] + (\beta-1)\ln\left(\frac{t_i + r\sum_{j=1}^{i-1}t_j}{\alpha}\right) + \frac{1}{(1-q)}\ln\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]\right\} \quad (16)$$

$$\text{s.t.} \quad (2-q) \geq 0, \quad (17)$$

$$1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta} \geq 0, \quad (18)$$

$$\frac{t_i + r\sum_{j=1}^{i-1}t_j}{\alpha} \geq 0, \quad (19)$$

$$1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta} \geq 0, \quad (20)$$

$$\alpha \geq 0, \quad (21)$$

$$\beta \geq 0. \quad (22)$$

### 3.4 *Particle Swarm Optimization*

PSO is a probabilistic optimization heuristic inspired by the social behavior of biological organisms, specifically the ability of animal groups to work as a whole in order to find some desirable position. This seeking behavior is artificially modeled by PSO, which has been mainly used in the quest for solutions of non-linear optimization problems in a real-valued search space [11].

For a problem with $n$-variables, each possible solution can be considered as a particle with a position vector of dimension $n$ and the population of particles is defined as swarm [12]. In PSO, particles move through the search space accordingly to the combination of the best solution they individually found and the best solution that any particle in the neighborhood found [11]. A neighborhood can be defined as the subset of particles with which a given particle is able to communicate. Each particle, represented by $j$, $j = 1, \ldots, n_{part}$ is composed by the following features: current position in the search space ($s_j$), best position it has visited so far ($p_j$), velocity ($v_j$) and fitness, which is the value of the objective function [11].

The q-Weibull-based GRP maximum likelihood problem, represented by the Equations (16) to (22), involves a four-dimensional search space where each dimension is related to the decision variables $\alpha$, $\beta$, $q$ and $r$. Therefore, $s_j, p_j, v_j$ are four-dimensional vectors that have their entries associated with $\alpha$, $\beta$, $q$ and $r$.

In this paper the lbest PSO algorithm is used, an approach that defines that a particle can obtain information only from a subset of particles, which is summarized in the flow chart presented in Figure 3. During the iterative process, the swarm evolution occurs as every particle has the velocity and position update equations applied to each dimension $k$ [9]:

$$v_{jk}(m+1) = \chi\left\{v_{jk}(m) + c_1 u_1\left[p_{jk}(m) - s_{jk}(m)\right] + c_2 u_2\left[p_{gk}(m) - s_{jk}\right]\right\}, \quad (23)$$

$$s_{jk}(m+1) = s_{jk}(m) + v_{jk}(m+1), \quad (24)$$

where $m$ is the iteration number, $\chi$ is the constriction factor which avoid velocity explosion during iterations, $c_1$ and $c_2$ are positive constants, $u_1$ and $u_2$ are independent uniform random numbers between 0 and 1, $p_{gk}$ is the $k$th entry of vector $p_g$ related to the best position that has been found by any neighbor of particle $j$ [9].

The updates of velocities and positions happen until a stop criterion is met. Three stop criteria are used in this paper: maximum number of iterations; the global best particle is the same for 10% of the maximum number of iterations; the global best fitness values in two consecutive iterations are different, but such a difference is less than a predefined tolerance δ.

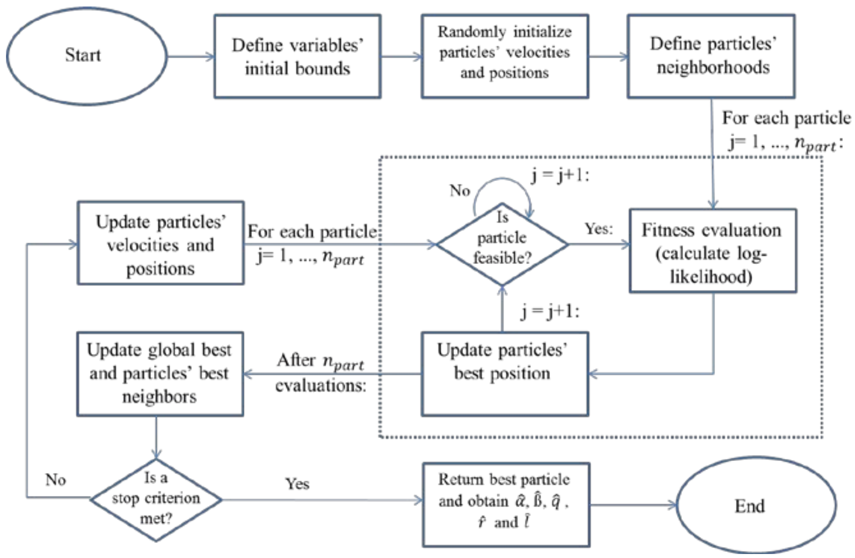


Figure 3 – PSO flow chart

## 4. RESULTS

The PSO algorithm was implemented in MATLAB and run in a personal computer with 2.53 GHz, 2Gb of RAM and Windows 7 operating system. The data for the application of PSO in order to obtain the maximum likelihood estimates for the q-Weibull-based GRP are presented in Table 2 and Table 3, which correspond to failures of a compressor [1] and of an NC machine tool [2], respectively. PSO parameters are listed in Table 4.

The estimated parameters of Yañez *et al.* [1] (compressor case) obtained from a GRP model based on the Weibull distribution were $\alpha = 3072$; $\beta = 1{,}620$, representing an increasing hazard rate and 0,70 as repair index, which suggest that the compressor's repair leads to a better than old but worse than new condition. The results of q-Weibull GRP model proposed in this paper, in turn, show an estimation of the repair index $r = 0{,}459$, a condition of better than old but worse than new, as the former result. However, the combination of $\beta = 2{,}519$ and $q = 1{,}139$ indicates a unimodal hazard rate (see Table 1), differently from the increasing behavior found by Yañez *et al.* [1] (Figure 4).

The estimated parameters for the NC machine tool case from Wang & Yang [2], which also uses a GRP based on the Weibull distribution, were $\alpha = 158{,}73$; $\beta = 0{,}766$ and $r = 0{,}109$ , which suggest a post-repair condition of better than old but worse than new and a monotonically decreasing hazard rate. The results obtained with q-Weibull of $\beta = 0{,}333$, $q = -2{,}659$ leads to a bathtub curve hazard rate (Figure 5) and $r = 0{,}022$ to the same repair effectiveness category as the work in comparison, although it is near to the perfect repair. These results are summarized in Table 5.

The unimodal behavior observed for the compressor in the q-Weibull GRP results can be justified as a case when a product has two or more failure modes or causes [13]. The second result, a bathtub curve for the NC machine tool, represents three distinct periods of the equipment failure behavior: the burn-in failure period; the period that the equipment presents approximately a constant failure rate; the wear-out failure period [13].

Table 2 – Time between failures for a compressor [1]

| Number of failures | Time between failures (h) | Number of failures | Time between failures (h) | Number of failures | Time between failures (h) |
|---|---|---|---|---|---|
| 1 | 3456 | 9 | 3072 | 17 | 244 |
| 2 | 1584 | 10 | 384 | 18 | 1528 |
| 3 | 236 | 11 | 2448 | 19 | 44 |
| 4 | 516 | 12 | 32 | 20 | 3064 |
| 5 | 1820 | 13 | 360 | 21 | 324 |
| 6 | 452 | 14 | 998 | 22 | 1528 |
| 7 | 432 | 15 | 656 | 23 | 348 |
| 8 | 1264 | 16 | 180 | 24 | 336 |

Table 3 – Time between failures for a NC machine tool [2]

| Number of failures | Time between failures (h) | Number of failures | Time between failures (h) | Number of failures | Time between failures (h) | Number of failures | Time between failures (h) |
|---|---|---|---|---|---|---|---|
| 1 | 27,51 | 8 | 341,4 | 15 | 76,43 | 22 | 432,42 |
| 2 | 340,01 | 9 | 9,28 | 16 | 471,23 | 23 | 87,75 |
| 3 | 27 | 10 | 88,17 | 17 | 32,4 | 24 | 81,01 |
| 4 | 1,12 | 11 | 86,34 | 18 | 86,43 | 25 | 220,05 |
| 5 | 11,11 | 12 | 318,44 | 19 | 83,18 | 26 | 91,7 |
| 6 | 25,74 | 13 | 323,12 | 20 | 196,27 | 27 | 82,17 |
| 7 | 81,68 | 14 | 169,63 | 21 | 70,91 | 28 | 92,98 |

Table 4 – PSO parameters

| Parameter | Value |
|---|---|
| Number of particles | 30 |
| Number of neighbors | 2 |
| Number of iterations | 10000 |
| Number of algorithm’s replication | 30 |
| $c_1 = c_2$ | 2,050 |
| $\chi$ | $7{,}298 \cdot 10^{-1}$ |
| $\delta$ | $10^{-16}$ |

PSO was replicated for both cases 30 times and the descriptive statistics and the maximum likelihood estimators for the best particle results are presented in Tables 6 and 7, respectively for the compressor case and the NC machine tool case. The computation time for the first case was, in average, 72 seconds per run, while the second case took approximately 395 per replication.

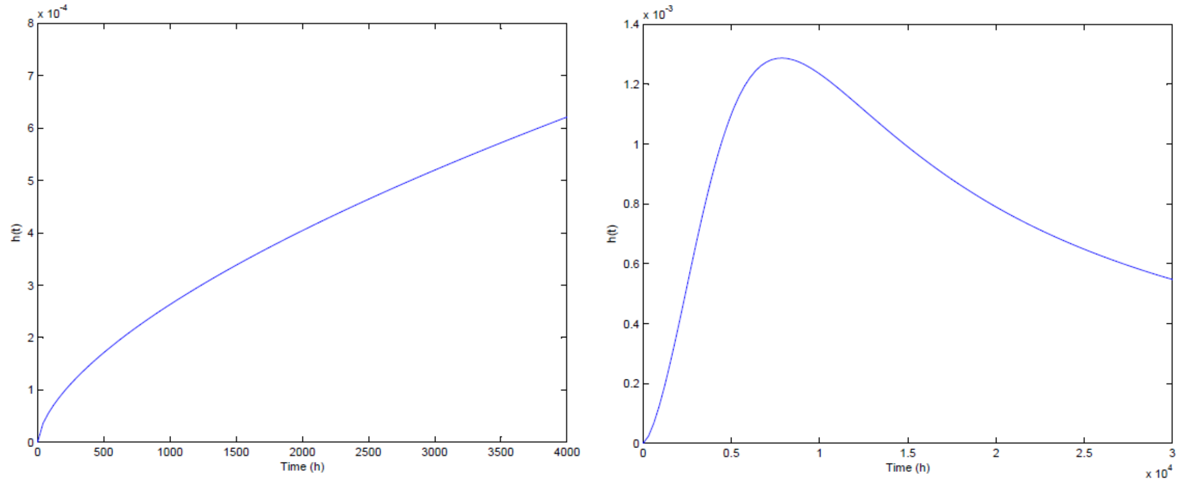


Figure 4 – Hazard rate functions related to the time until first failure occurrence from Weibull-based [1] and q-Weibull-based (this work) GRPs, compressor case

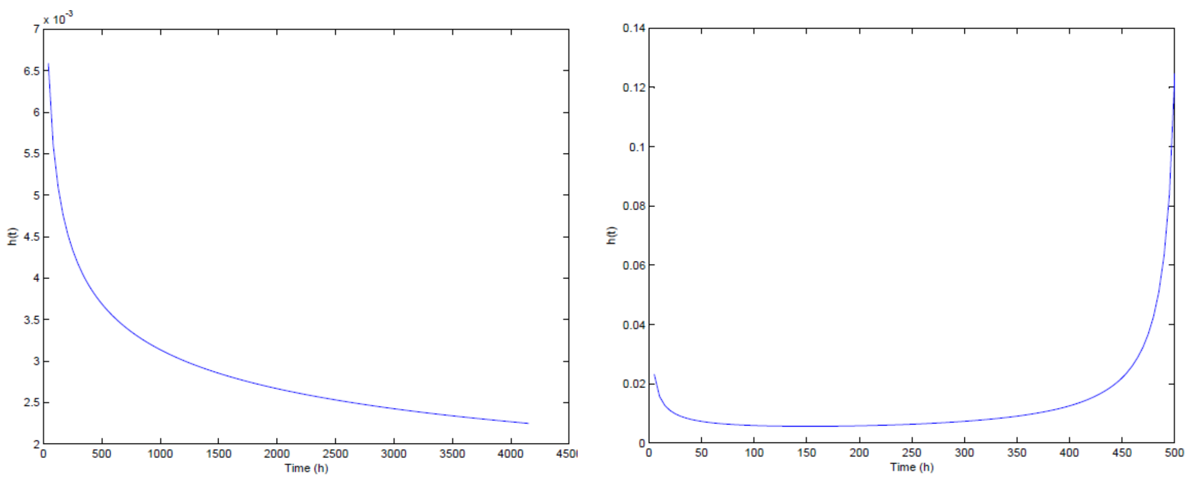


Figure 5 – Hazard rate functions related to the time until first failure occurrence from Weibull-based [2] and q-Weibull-based (this work) GRPs, NC machine tool case

Table 5 – Comparison between Weibull and q-Weibull GRP

| | | Compressor | | NC Machine tool | |
|---|---|---|---|---|---|
| | | **Weibull GRP** | **q-Weibull GRP** | **Weibull GRP** | **q-Weibull GRP** |
| **Parameters** | $\alpha$ | 3072 | 2954,760 | 158,730 | 26021,081 |
| | $\beta$ | 1,620 | 2,510 | 0,766 | 0,333 |
| | $q$ | - | 1,130 | - | -2,659 |
| | $r$ | 0,700 | 0,460 | 0,109 | 0,022 |
| **Hazard Rate** | | Monotonically increasing | Unimodal | Monotonically decreasing | Bathtub curve |

The differences between the results of q-Weibull-GRP and Weibull-GRP presented may be justified by the higher flexibility of the q-Weibull model when compared to the traditional Weibull distribution. Therefore, the proposed model may represent an efficient alternative modeling approach to repairable systems, being able to incorporate different behaviors of the hazard rate function, including the unimodal and bathtub-shaped ones.

Table 6 – PSO results for compressor case

| **Compressor** | | **Best particle** | **Minimum** | **Maximum** | **Median** | **Mean** | **Std. Dev** | **Coefficient of variation** |
|---|---|---|---|---|---|---|---|---|
| **Parameters** | $\alpha$ | 2954,76151 | 2954,76122 | 2954,76171 | 2954,76147 | 2954,76147 | 1,23E-04 | 4,17E-08 |
| | $\beta$ | 2,51957 | 2,51957 | 2,51957 | 2,51957 | 2,51957 | 1,83E-07 | 7,25E-08 |
| | $q$ | 1,13933 | 1,13933 | 1,13933 | 1,13933 | 1,13933 | 3,06E-08 | 2,69E-08 |
| | $r$ | 0,45855 | 0,45855 | 0,45855 | 0,45855 | 0,45855 | 4,94E-08 | 1,08E-07 |
| **Maximum Log-Likelihood** | | -189,04136 | -189,04136 | -189,04136 | -189,04136 | -189,04136 | 3,77E-14 | 1,99E-16 |

Table 7 – PSO results for NC machine tool case

| **NC Machine Tool** | | **Best particle** | **Minimum** | **Maximum** | **Median** | **Mean** | **Std. Dev** | **Coefficient of variation** |
|---|---|---|---|---|---|---|---|---|
| **Parameters** | $\alpha$ | 26021,08099 | 2346,13792 | 26021,08099 | 24798,29357 | 24827,15365 | 6,28E+02 | 2,53E-02 |
| | $\beta$ | 0,33342 | 0,33337 | 0,33867 | 0,33516 | 0,33533 | 1,15E-03 | 3,42E-03 |
| | $q$ | -2,65940 | -2,68131 | -2,56253 | -2,62673 | -2,62921 | 2,83E-02 | 1,08E-02 |
| | $r$ | 0,02206 | 0,02155 | 0,02215 | 0,02184 | 0,02183 | 1,47E-04 | 6,74E-03 |
| **Maximum Log-Likelihood** | | -164,52931 | -164,53395 | -164,52931 | -164,53145 | -164,53140 | 1,19E-03 | 7,25E-06 |

## 5. CONCLUSIONS

In this paper a maximum likelihood estimation for the failure-terminated case of q-Weibull GRP using Kijima virtual age type I parameters is proposed. Although the Weibull distribution has been widely used with GRP, the q-Weibull is expected to bring more flexibility to the model because of the $q$ parameter affecting the shape as well as $\beta$ parameter.

Due to the complexity of solving the maximum likelihood optimization problem, a probabilistic heuristic is used instead of solving the corresponding system of derivatives. The chosen method, PSO, proved to be a great tool for solving non-linear optimization. In fact, the PSO provided coherent estimates for the parameters of the q-Weibull GRP. The results found in this paper were compared with two papers of Weibull GRP applications and differences between the hazard rate's behaviors were observed. The numerical experiments provided low standard deviations for the parameter estimates and also for the maximum log-likelihood values, which indicates the PSO ability in providing very similar solutions for the q-Weibull maximum likelihood problem related to a specific failure data set in different runs.

A step forward to this paper is to formulate the maximum likelihood estimators of q-Weibull GRP for the time-terminated cases and combine the PSO algorithm to local search methods in order to enhance its robustness with respect to the q-Weibull-GRP log-likelihood problem.

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