Reliability Analysis of a Sugarcane Harvester using q-Weibull distribution

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Abstract: In this work, q-Weibull distribution is applied to analyze the reliability of a sugarcane harvester used in a process of mechanized cutting of sugarcane, based on historical time-to-failure data. It is compared to the original three-parameter Weibull distribution, highlighting the advantages of this model. The parameters were estimated by the least square estimation (LSE) method. The fitting results of the q-Weibull distribution provided a coefficient of determination (R^2) higher than that value obtained with Weibull distribution. The better performance of q-Weibull distribution was also confirmed by visual inspection once it exhibits a nonmonotonic failure rate, which is impossible to obtain with the three-parameter Weibull distribution. Results also show that q-Weibull performs better than the three-parameter Weibull distribution and is able to represent the entire life cycle more accurately.

Keywords: Reliability; q-Weibull distribution; Failure rate; Bathtub curve; Sugarcane harvester

1. INTRODUCTION

One of the main tasks of reliability engineering is to keep industrial plants, equipment, or simple components functioning properly. A deep assessment for understanding failure prevention is related to quantitative models that demand a systematic statistical approach to the description of the failure rate of components, equipment and systems. The choice and application of a model that accurately characterizes the failure become the aim and the most important step in reliability analysis [1]. The three-parameter Weibull distribution is one of the most common distributions applied in reliability analysis and has been widely used in many industrial applications, such as automotive, aerospace, military, nuclear power, electronics, electrical power, advertising, dental research and the mortality of mailing lists [2-4]. This Weibull distribution is also indicated to evaluate the reliability of components that suffer wear-out failure and to determine the optimum replacement or repair interval [5].

The survival probability density function (pdf) at time t described by Weibull in Ref. [2], can be written as:

$$f(t) = \frac{\beta}{\eta - t_0}\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta-1} \exp\left[-\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta}\right], \tag{1}$$

where t is the time-to-failure or lifetime, β is the shape parameter, $\eta - t_0$ (known as θ) is the scale parameter and t_0 is the location or minimum life parameter, with $\beta > 0$, $\eta - t_0 > 0$ and $t \geq t_0$. When $\beta = 1$, the exponential distribution is obtained. The characteristic life parameter η is defined as the time value life $t = \eta$ at which 63,2% of units will fail. η and t_0 have the same unit of time t. β is dimensionless.

Despite the simplicity and the numerous applications, the Weibull distribution has some limitations. The q-Weibull distribution is a four-parameter generalization which interpolates the q-exponential and Weibull ones, and has been studied and applied in some areas of engineering, presenting a better performance than the usual three-parameter Weibull distribution. One of the first applications of the q-Weibull distribution in reliability analysis, i.e. modeling the time-to-failure due to the dielectric rupture of oxides in electronic devices, provided a better quality fit than that obtained by Weibull [6]. A similar study of the fit of time-to-failure data from a natural gas recovery plant was performed by Sartori *et al.* [7] and also showed the superiority of q-Weibull due to the additional q parameter. In a comparative study between the q-exponential and Weibull distributions for highway length, it was found that the q-exponential and Weibull distributions do not give a satisfactory adjustment, being necessary to employ the q-Weibull distribution [8].

In addition, the usual Weibull model is not able to represent the entire life cycle of an asset with a single function, since it only expresses monotonous failure rates. Mathematical properties of the q-Weibull model which were explored and led up by Assis, Borges and Melo [9], opened a new field of research that had not been covered earlier in the literature. They showed that the q-Weibull model can also display, besides the monotonic curves, nonmonotonic failure rate form: the well-known bathtub curve (U-shaped) and the unimodal shape. Each type of failure rate behavior has a specific range of value of shape parameters q and β. They continued with this study and showed that for modeling bathtub curve, the original Weibull model requires three functions, one for each failure rate segment decreasing, constant and increasing, respectively represented by the values of the shape parameter $\beta < 1$, $\beta = 1$ and $\beta > 1$, and confirmed again that the generalized q-Weibull is capable to model in addition to the monotonous failure rates, the other two types of failure rate (bathtub curve and unimodal), both being reproduced with a single set of parameters [1]. They also compared four models for the description of lifetime of a robotic welding station used in a manufacturing process (the exponential, the Weibull, the q-exponential and the q-Weibull). Two of these models are generalized versions of the usual ones. The results show that the q-Weibull model is more flexible to describe shapes of failure rate curves than the other models. Other generalizations and modifications of the Weibull distribution can be found in the literature (see Ref. [10, 11]).

It is important to note that the q-Weibull distribution has been connected with the dimensionless entropic parameter q in the context of Tsallis statistics. The Tsallis pioneering paper has introduced a generalization of the concept of entropy [12]:

$$S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \ (q \in \mathbb{R}), \tag{2}$$

where k is a conventional positive constant, W is the total number of possible configurations and p_i is the associated probabilities. At the limit $q \to 1$, the Boltzmann-Gibbs statistics is recovered $S_1 = -k\sum_{i=1}^{W} p_i \ln p_i$. For a recent review see Ref. [13].

In order to apply the q-Weibull distribution in reliability modeling, some others important mathematical functions which were generalized in the context of nonextensive statistical mechanics, need to be considered. Tsallis also defined the q-logarithm and its inverse, the q-exponential [14]:

$$\ln_q(x) = \frac{x^{1-q}-1}{1-q} \ (x > 0), \tag{3}$$

$$\exp_q(x) = \begin{cases} [1+(1-q)x]^{(1/(1-q))}, & \text{if } (1+(1-q)x) > 0 \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

in which $x, q \in \mathbb{R}$. At the limit $q \to 1$, these functions recover the usual logarithm $\ln_1(x) = \ln(x)$ and exponential $\exp_1(x) = \exp(x)$. The cut off condition presented in Equation (4) avoid negative and complex numbers, in order to achieve the probabilities. The q-exponential change from exponential behavior (with $q = 1$) for asymptotic power law with large x and $q > 1$ presenting stretched tail. More details about these mathematical operations can be viewed in Ref. [8, 9].

The purpose of this paper is to apply the q-Weibull distribution to analyze the reliability of a sugarcane harvester currently in operation, based on historical time-to-failure data. Previous papers gave fundamentals support to recall q-Weibull model and to analyze some details that are important to reliability analysis, such as Ref. [1, 6-9, 12, 14].

In the following section, we present the generalized q-Weibull distribution with mathematical equations and the parameter estimation. The application of this distribution in a sugarcane harvester is considered in Section 3. The last section is dedicated to our conclusions.

2. *q*-WEIBULL DISTRIBUTION

Using the definition of the q-exponential for the generalization of the Weibull distribution (Equation (1)), the probability density function of q-Weibull, with $t \geq t_0$, can be written as (see Ref. [6] for interpretation):

$$f_q(t) = (2-q)\frac{\beta}{\eta - t_0}\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta-1} \exp_q\left[-\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta}\right]. \tag{5}$$

The constraint $q < 2$ and the factor $(2-q)$ are necessary to ensure the normalization of $f_q(t)$.

The following q-Weibull expressions was discussed and applied in Ref. [9]. The reliability function is defined by:

$$R_q(t) = \left\{\exp_q\left[-\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta}\right]\right\}^{2-q}. \tag{6}$$

The probability of failure $F_q(t)$ defines the cumulative fraction of parts that will fail by a time t:

$$F_q(t) = 1 - R_q(t) = 1 - \left\{\exp_q\left[-\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta}\right]\right\}^{2-q}. \tag{7}$$

And so the failure rate function is:

$$h_q(t) = \frac{f_q(t)}{R_q(t)} = \frac{(2-q)\beta}{\eta - t_0}\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta-1}\left\{\exp_q\left[-\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta}\right]\right\}^{q-1}. \tag{8}$$

The failure rate becomes the original Weibull when $q = 1$:

$$h(t) = \frac{\beta}{\eta - t_0}\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta-1}. \tag{9}$$

The failure rate behavior is one of the main discussion we analyze here. The difference between Weibull and q-Weibull can be seen by comparing Equations (8) and (9) and the visual shape of each model is being shown in Figure 1, depending on the values of the shape parameters β and q. Table 1 shows the possibilities of these behaviors. Although not shown in Figure 1, note that the function also reproduces the constant failure rate for $q = 1$ and $\beta = 1$.

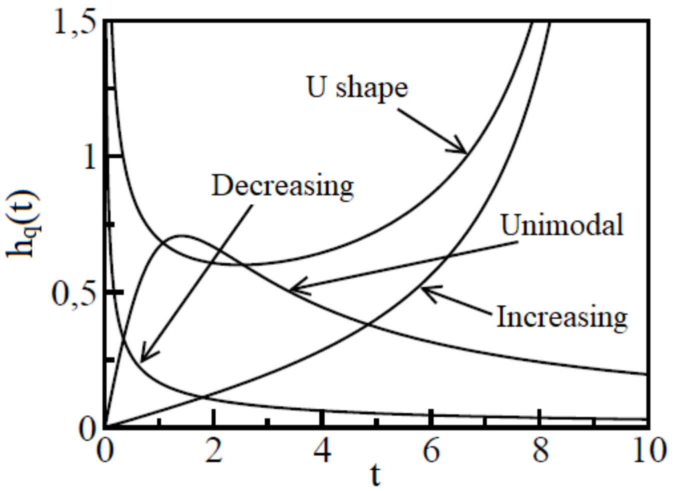

Figure 1 - Four different types of behavior of the failure rate function.
Source: Assis *et al*. (2013)

Table 1 - Behavior of the q-Weibull failure rate according to the values of the parameters q and β

	$0 < \beta < 1$	$\beta = 1$	$\beta > 1$
$q < 1$	Bathtub Curve	Monotonous increasing	Monotonous increasing
$q = 1$	Monotonous decreasing	Constant	Monotonous increasing
$1 < q < 2$	Monotonous decreasing	Monotonous decreasing	Unimodal

Source: Assis *et al*. (2013)

For the estimation of parameters, the sample data which are time-to-failure must be in ascending order. The median rank is the most popular approach of estimating the Y-axis plotting positions and regression analysis to fit the line. Weibull employed mean ranks in his paper but later, Johnson became recognized suggestions to use median ranks by means of Bernard`s approximation, an adjusted median rank [2, 15]. Some examples using Bernard's approximation for the median rank were taken and demonstrated that is sufficiently accurate for plotting and estimating the parameters. It is also easier than interpolating in the tables for the adjusted median ranks that are not an integer value [1, 4, 7, 9]. So, an estimative of unreliability can be done:

$$\hat{F}_i = \frac{i - 0{,}3}{n + 0{,}4}, \tag{10}$$

where i is the failure order number which ranges from 1 to n and n is the sample size. Note that if two data points have the same time to failure on the X-axis, they are plotted at different median rank values on the Y-axis, each point gets its own individual vertical location. For each sampling time t_i, we have:

$$x_i = \ln(t_i - t_0), \tag{11}$$

$$y_i = \ln\left[-\ln_{q'}\left(1 - \hat{F}_i\right)\right]. \tag{12}$$

Equation (7) can be described as $y = \beta x + b$, placing the sample data in a straight line by the change of variables x_i and y_i, represented by Equations (11) and (12) and $b = -\beta \ln\left[(\eta - t_0)/(2-q)^{1/\beta}\right]$ (see Ref. [1, 6] for details). Realize that there is a q-logarithm in y_i expression, denominated $q' = 1/(2-q)$, so the common procedure to calculate y_i for q-Weibull distribution must be changed, considering the generalized mathematical functions in the context of nonextensiveness mentioned above in Equations (3) and (4). We obtain the graph of $\ln\left[-\ln_{q'}\left(1-\hat{F}\right)\right]$ versus $\ln(t - t_0)$ since $\ln\left[-\ln_{q'}\left(1 - F_q(t)\right)\right] = \beta \ln(t - t_0) - \beta\ln\left[(\eta - t_0)/(2-q)^{1/\beta}\right]$, as described in Ref. [8].

The parameters of q-Weibull distribution (β, t_0, η and q) are estimated via the least squares estimation (LSE) method, maximizing the coefficient of determination, searching the parameters q and t_0 that return the maximum value of R^2, which represents the quality of the fit [7]:

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}, \tag{13}$$

where the adjustment curve of the model $\hat{y}_i$ is $\ln\left[-\ln_{q'}\left(1 - F_q(t_i)\right)\right]$ and the mean $\bar{y}$ is $\sum y_i/n$. The constraints are $\beta > 0$, $\eta > t_0$, $\theta > 0$, $t_0 < t_{min}$ and $q < 2$. t_{min} is the lowest sample time. Note that the parameters of the original Weibull distribution can be obtained by imposing a constraint $q = 1$ on the four-parameter q-Weibull distribution. Equation (13) return $R^2 \leq 1$, including negative values.

Other methods of parameter estimation were proposed. The maximum likelihood estimation (MLE) method was used in a detailed study by Jose and Naik [16] and showed the properties of the q-Weibull distribution in applications to a data on cancer remission times. The results showed that the q-Weibull model has a better fit than Weibull, but they claimed the difficult to estimate the parameters due to the nonlinear set of equations. Other attempts with the MLE method were also performed and showed some difficulties with original Weibull distribution. It was cleared that the calculation is difficult and iterative for the Weibull parameters and so convergence may not always occur. It was also shown that it is satisfactory particularly for large samples over 500 failures [4]. However, another way of estimating parameters has been presented using also the likelihood function, despite the difficulty to converge and it was applied with the fresh generalized

q-Weibull model discussed here. To overcome this problem, Xu *et al*. [17] proposed recently an adaptive hybrid artificial bee colony algorithm denominated AHABC. The parameter estimation procedure was applied to real reliability failure data and showed its effectiveness by producing a more accurate convergence.

Despite the good results presented by Xu *et al*. [17] in dealing with non-trivial parameter estimation due to the intricate system of nonlinear equations, showing that the algorithm efficiently finds the optimal solution for the q-Weibull MLE problem, the maximum likelihood method was not used in our study.

3. APPLICATION TO A SUGARCANE HARVESTER

In this section, we apply the q-Weibull distribution in comparison to the three-parameter Weibull model, to analyze the time-to-failure data of a sugarcane harvester from the historic industrial database. This machine has seven systems previously hierarchized: drive, feeding, straw extraction, transport of sugarcane, propulsion, hydraulic and electric. Particularly, we analyze the systems of the transport of sugarcane and the propulsion, in which have respectively, 71 and 43 operation times, in hours. There is no censored data in the samples. All values of the samples were used to estimate the unreliability, according to the x_i and y_i variables and median ranks.

Table 2 shows the fitting parameters for each system and the coefficient of determination R^2.

Figures 2 and 4 show the usefulness of the q-Weibull distribution in comparison with usual Weibull model ($q = 1$), when the models fit the points in a straight line, in a graph $\ln\left[-\ln_{q'}\left(1-\hat{F}\right)\right]$ versus $\ln(t - t_0)$. In a qualitative inspection, note that q-Weibull fits better the data, especially in the tails in both systems analyzed. Thus, the q-Weibull presented a superior coefficient of determination.

Table 2 - Fitting results

System	Model	β	η (hour)	t_0 (hour)	θ (hour)	q	R^2
Transport of sugarcane	q-Weibull	0.59	9924.22	1.36	9922.86	-6.43	0.9847
	Weibull	0.85	112.15	0.46	111.69	1.00	0.9686
Propulsion	q-Weibull	1.67	204.60	-5.53	210.13	1.36	0.9860
	Weibull	1.23	344.24	7.17	337.07	1.00	0.9792

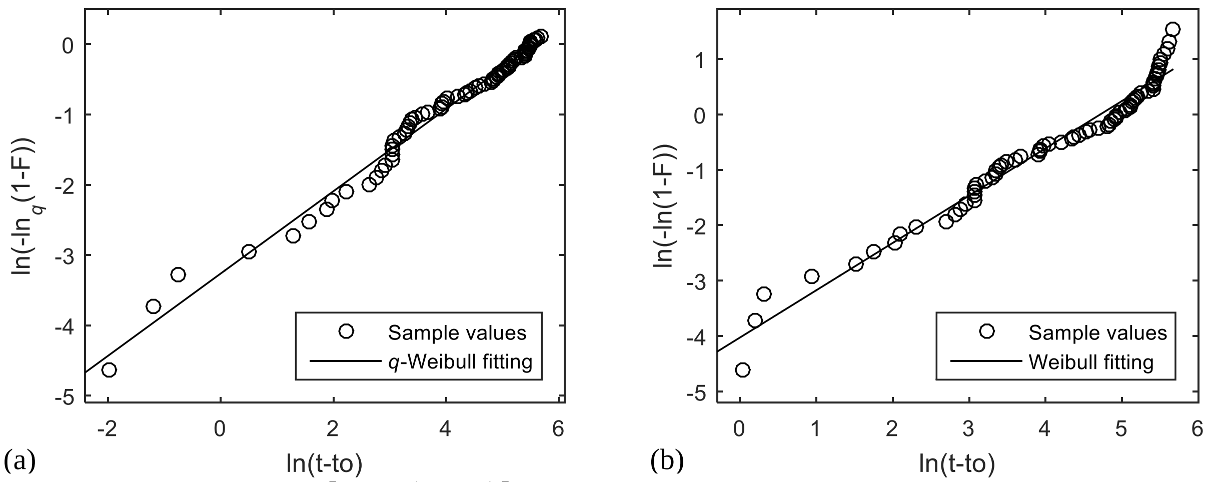

Figure 2 - (a) Graph of $\ln\left[-\ln_{q'}\left(1-\hat{F}\right)\right]$ versus $\ln(t - t_0)$ of the transport of sugarcane system lifetime. Parameter $q = -6.43$ and square correlation coefficient is $R^2 = 0.9847$; (b) the same graph, data and system with parameter $q = 1$ and square correlation coefficient $R^2 = 0.9686$.

Figures 3 and 5 show the reliability and failure rate curves for the two systems. Panel (a) shows another way to evaluate the fitness of the models, comparing the reliability curves with the experimental data (circles). It can be noted again that the q-Weibull is able to fit better all the range of the data, while the original Weibull model constantly diverges from experimental data in both cases, especially in large time-to-failure like shown in Figures 2 and 4. Notice that, as the lifetime increases, the curves of the two distributions move away and q-Weibull gets closer to the samples. This characteristic makes this model most appropriate for these sample data. The experimental data are the unreliability estimates $\hat{F}_i$ (Equation 10) versus time-to-failure in log-log plot of reliability curves.

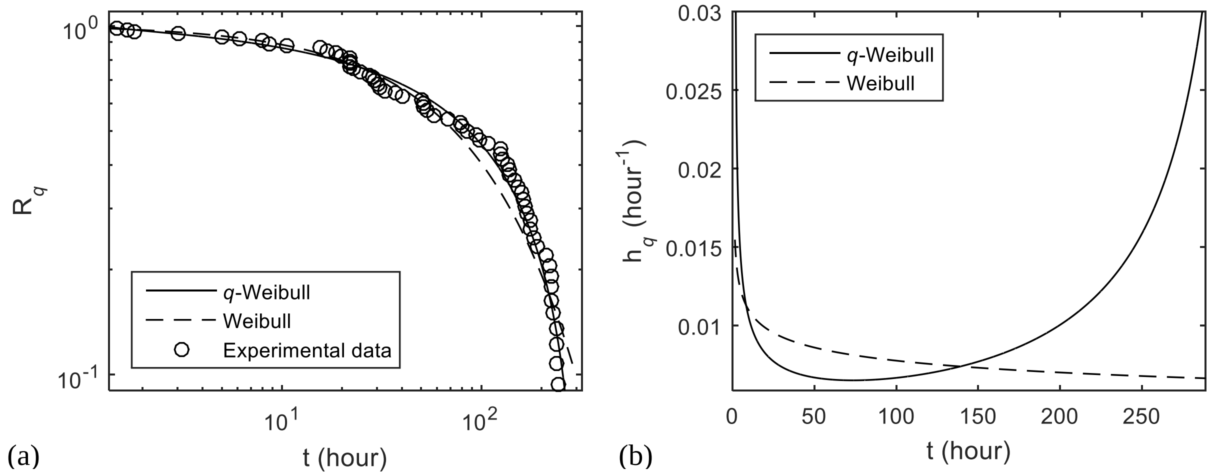

Figure 3 - (a) Reliability curves and experimental data; (b) failure rate curves; both plots show time-to-failure of the transport of sugarcane system on the X-axis.

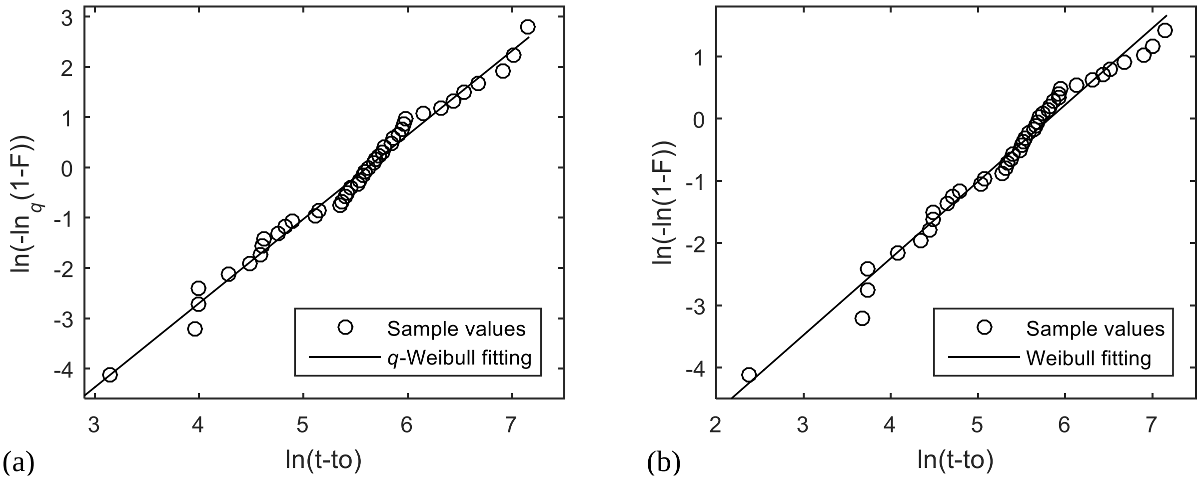

Figure 4 - (a) Graph of $\ln\left[-\ln_{q'}\left(1-\hat{F}\right)\right]$ versus $\ln(t-t_0)$ of the propulsion system lifetime. Parameter $q = 1.36$ and square correlation coefficient is $R^2 = 0.9860$; (b) the same graph, data and system with parameter $q = 1$ and square correlation coefficient $R^2 = 0.9792$.

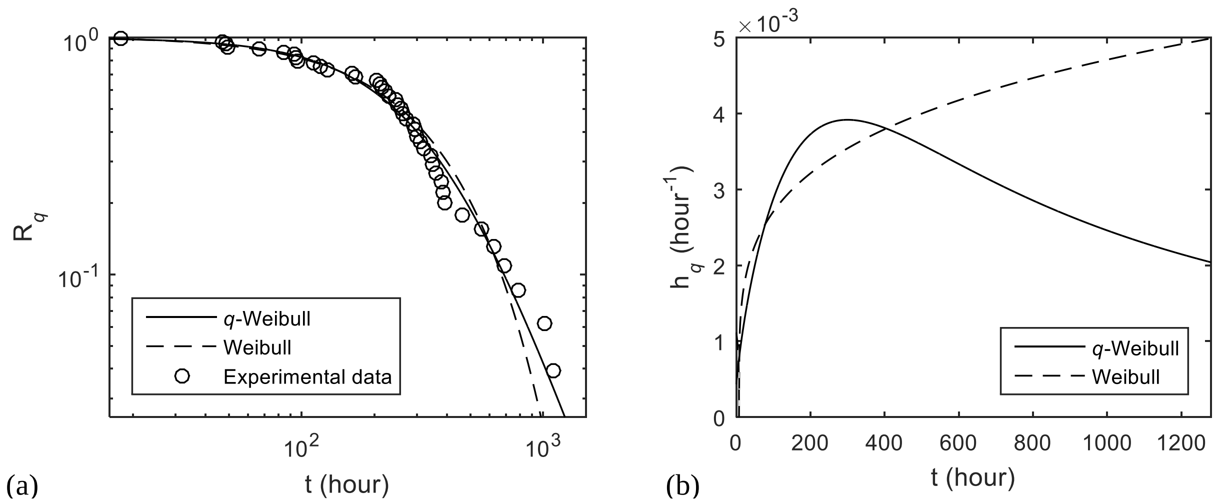

Figure 5 - (a) Reliability curves and experimental data; (b) failure rate curves; both plots show time-to-failure of the propulsion system on the X-axis.

The shape of the failure rate (life cycle) of the q-Weibull is obtained through Equation (8), while to obtain the original Weibull failure rate form, simply making $q = 1$ (panel (b)). The examples show different behaviors of the failure rate $h_q(t)$ including nonmonotonic forms, according to the values of the shape parameters q and β. The first work recorded in the literature that reached such a representation of the bathtub curve and unimodal shape with the q-Weibull distribution is found in Ref. [9], in which it provided resources for the results presented here. Table 3 shows the behavior of failure rate for each system analyzed. Another possibilities combinations of parameters and their related behaviors were shown in Table 1.

Table 3 - Behavior of q-Weibull failure rate results for each system according to the range of the shape parameters q and β

System	Model	q range	β range	Failure rate behavior
Transport of sugarcane	q-Weibull	$q < 1$	$0 < \beta < 1$	Bathtub shape
	Weibull	$q = 1$	$0 < \beta < 1$	Monotonous decreasing
Propulsion	q-Weibull	$1 < q < 2$	$\beta > 1$	Unimodal
	Weibull	$q = 1$	$\beta > 1$	Monotonous increasing

4. CONCLUSIONS

In several application areas of reliability engineering, the Weibull and other generalized models for lifetimes have an important role to describe many frequency distributions. In this work, we compare the classical three-parameter Weibull distribution with the generalized four-parameter q-Weibull distribution. In order to investigate the benefits of an additional parameter, both models were applied to describe life data of a sugarcane harvester, in a specific mechanized cane cutting process.

Bernard's approximation of the median ranks was used as well as the least squares estimation (LSE) method to estimate the parameters of the models, maximizing the coefficient of determination R^2. The results show that the q-Weibull distribution performs better than the classical three-parameter Weibull, for the two examples addressed (the systems of transport of sugarcane and propulsion). The q-Weibull model fits the sample data better than the Weibull model and presents, therefore, an improvement to describe events, since the prediction of failures can be obtained with greater precision. Such an improvement was expected due to the additional parameter q, but it is important to also note the ability of the q-Weibull model to describe other two nonmonotonic failure rate behaviors, which have been shown in the examples here: the bathtub curve (or U-shaped) and the unimodal shape, in which the original Weibull model is unable to describe, once is monotonically decreasing, constant or monotonically increasing, depending on the value of parameter β. These three phases are also known as infant mortality, useful life, and wear out. As it can be seen in Assis *et al*. [1], for bathtub curve modeling, original Weibull model requires three functions, one for each failure rate segment decreasing, constant and increasing, respectively represented by the range of the β shape parameter $0 < \beta < 1$, $\beta = 1$ and $\beta > 1$. These three functions indeed demand more time and a larger set of parameters, while q-Weibull describes nonstop the entire bathtub curve maintaining the same set of four parameter values.

Since the useful q-Weibull distribution reduces the time for parameter estimation and can reproduce continuously the whole life cycle of an item with a variety of failure rate behaviors, its flexibility and accuracy in fit will further enhance even more the operational process industry reliability modeling.

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