**Application of the Penalized *q*-Exponential Function in Reliability Datasets**

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**ABSTRACT**

In reliability field we use some distributions to obtain failures information as the expected number of failures, conditional reliability, hazard rate, etc. However, to obtain reliable system information we should use a distribution that can fit well the observed system. In addition, in reliability area, we are also interested in model data in different phases of the bathtub curve. The *q*-Exponential distribution is able to model data in the three phases of this curve, however in the last phase was verified a possible problem called “monotone likelihood”. In this paper we apply the *q*-Exponential distribution corrected by the Firth’s and resample methods to fit four datasets in the wear-out phase, which corresponds to the last phase of the bathtub curve. We find that for the two of the datasets we should use the *q*-Exponential to fit the data, but in one case it is not necessary apply the corrected function. The Weibull model was the worst in both situations.

**Keywords:** Expected number of failures; *q*-Exponential distribution; Wear-out phase; Bathtub curve; Weibull model.

1. **INTRODUCTION**

Reliability area is interesting in provide ways to compute important measures about many of systems. In general, these measures can provide prevent and/or solutions information. In this way, probabilistic distributions have been used to help in this issue. There are some distributions that have been very used in reliability field, such as Weibull and Exponential. However, over time other models emerged as good alternative to these and others models. In particular, the q-Exponential distribution is one of that, and it is able to model the three phases of the bathtub curve. Thus, it started to be a important distribution to the reliability field.

However, the *q*-Exponential model, which has the power law behavior, present a problem of estimation when , as can be seen in [1]. This problem is called “monotone likelihood” and when it is verified the methods to estimate the parameters can fail when .

Besides reliability, the *q*-Exponential has been applied to a variety of problems in many research areas. For example, [2] showed that the population of a country is well described by a *q*-Exponential distribution with PDF presenting a power law behavior. In [3]was verified that the temporal correlation function of hydrogen bonds can be modeled by a *q*-Exponential probabilistic model. The authors in [4] introduced a *q*-Exponential regression model for fitting data with discrepant observations. In addition, [5] used the *q*-Exponential model into financial market field; they showed that this distribution can reproduce the complex dynamic behavior of the markets. In reliability, [6] used the *q*-Exponential to infer about a useful performance metric in system reliability, the index , where is the stress, is the strength and both are supposed independent *q*-Exponential random variables with different parameters. Besides, [7] developed a *q*-Exponential generalized renewal process (GRP) and a *q*-Weibull-GRP. The authors apply both models to fit failure data of complex systems and *q*-Exponential-GRP outperformed the Weibull-GRP approach.

In this paper we will use the function corrected as in[1]and corrected by the resample method to fit four reliability datasets. Besides, we will to make comparisons with the original *q*-Exponential and Weibull models.

The rest of the paper unfolds as follows. Section 2 has a description of the work realized. Section 3 brings the *q*-Exponential distribution and the presentation of the used methods to correct the *q*-Exponential distribution. Section 4 present two example applications. Section 5 brings a discussion about obtained results and Section 6 present the conclusion of the work.

1. **DESCRIPTIONOF WORK REALIZED**

This work was conducted to apply a methodology for the corrected *q*-Exponential function; thus, it is characterized as an applied research. This paper presents five empirical applications of the referred corrected function.

Besides, this work can be classified as qualitative and quantitative. Is is quantitative because use statistical models and computational software to achieve the required results. On other hand, it is qualitative because apply concepts of the literature to understand and analyze the obtained results.

This research can be sub-divided in the following steps:

* Step 1: we search in the literature reliability data sets in the wear-out phase to apply the original and corrected *q*-Exponential log-likelihood functions and compare the results with the Weibull model.
* Step 2: Then, we applied the original *q*-Exponential, the corrected *q*-Exponential log-likelihood functions and the Weibull distributions to the datasets.
* Step 3: Next, we made comparisons between the functions.

1. **THEORETICAL BACKGROUND**

3.1. *q-Exponential Distribution and Correction Methods*

The *q*-Exponential distribution has the following PDF:

, (1)

where determines the PDF shape and is known as entropic index, while is the scale parameter. In the limit , Equation (1) recovers the Exponential distribution. When , Equation (1) has a limited support with an upper bound that depends on and ; see Equation (2).

. (2)

For the sake of illustration, Figure 1(a) shows the behavior of the *q*-Exponential PDF for , and three possible values of , and Figure 1(b) presents the *q*-Exponential PDF for , and three possible values of . Note, in Figure 1(b), that when and the support is limited by .

|  |  |
| --- | --- |
|  |  |

Figure 1 -*q*-Exponential PDF a) for and some possible values of ; b) for and some possible values of *q.*

The *q*-Exponential has the following Cumulative Distribution Function (CDF):

. (3)

By definition, the hazard rate is [8], where is the reliability function with . Thus, it follows that:

. (4)

The *q*-Exponential hazard rate can be monotone increasing, monotone decreasing or constant for , and , respectively. In fact, this is an important characteristic of the *q*-Exponential distribution, especially in the reliability context because it enables modelling each of the three phases of the bathtub curve as Weibull model does. Figure 2 presents examples of increasing and decreasing hazard rates.

|  |  |
| --- | --- |
|  |  |

Figure 2 -*q*-Exponential hazard rate a) for and ; b) for and *.*

In order to generate pseudorandom numbers that follow a *q*-Exponential distribution, Equation (3) can be used; it is obtained by means of the inverse transform method [9]:

, (5)

where denotes a uniform pseudorandom number.

In this work, our purpose is showing how good is the results provided by the penalized functions and compare the results with the original *q*-Exponential and Weibull models. Thus, we will only summarize how the method were applied to the *q*-Exponential model, for more details about the Firth and resample methods see [10, 11], respectively.

The correction of the likelihood function is applied as following

, (6)

where refers to the determinant of the Fisher information matrix, and the penalization term is the Jeffreys invariant prior [12]. Then, by applying the logarithm in Equation (6), parameter estimation can be executed by maximizing

. (7)

To obtain the estimates of the *q*-Exponential distribution by the resample method we must rewrite the log-likelihood in Equation (8) as a function of as follows

. (8)

Then, can be obtained by replacing by . The is given by

. (9)

Where is a vector of the bootstrap samples described by the weight that each observation receives in the new empirical distribution function.

1. **OBTAINED RESULTS**

4.1. *Example Application 1*

The first application example, whose data are in Table 1, involves a magnetic resonance imaging (MRI) equipment. MRI scanners use strong magnetic fields, radio waves, and field gradients to generate images of the organs in the body to be analyzed by doctors and specialists [13].

For the two application examples presented in this work the parameter estimates were obtained by the original *q*-Exponential log-likelihood function and by the functions penalized by the Firth’s and resample methods. Besides, also considering all the application examples, The Nelder-Mead optimization method was used, and the initial parameters were set to and for original and corrected functions.

Table 1 - TBFs of the MRI scanner (in days)

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **1** | 99 | **14** | 6 | **27** | 54 | **40** | 140 | **53** | 46 |
| **2** | 38 | **15** | 78 | **28** | 22 | **41** | 19 | **54** | 17 |
| **3** | 109 | **16** | 77 | **29** | 13 | **42** | 10 | **55** | 7 |
| **4** | 10 | **17** | 24 | **30** | 54 | **43** | 17 | **56** | 75 |
| **5** | 35 | **18** | 66 | **31** | 19 | **44** | 4 | **57** | 58 |
| **6** | 42 | **19** | 25 | **32** | 47 | **45** | 54 | **58** | 102 |
| **7** | 31 | **20** | 4 | **33** | 14 | **46** | 26 | **59** | 6 |
| **8** | 18 | **21** | 8 | **34** | 53 | **47** | 135 | **60** | 53 |
| **9** | 53 | **22** | 26 | **35** | 14 | **48** | 44 | **61** | 47 |
| **10** | 3 | **23** | 98 | **36** | 35 | **49** | 59 | **62** | 26 |
| **11** | 12 | **24** | 11 | **37** | 73 | **50** | 11 | **63** | 87 |
| **12** | 13 | **25** | 87 | **38** | 18 | **51** | 18 | **64** | 6 |
| **13** | 40 | **26** | 11 | **39** | 38 | **52** | 3 | **65** | 13 |

Table 2 -Parameter estimates and log-likelihood values – Example 1

|  |  |  |  |
| --- | --- | --- | --- |
| Method |  |  | Log-likelihood value |
| Original | 0.7122 | 60.4849 | -296.0661 |
| Firth’s | -2.5598 | 498.3812 | -289.4815 |
| Resample | -0.7391 | 248.5461 | -311.874 |

Table 2 shows the results of the estimation for the tree log-likelihood function: the original and the penalized ones. We can see in Table 2 that the Firth’s method provided the greater log-likelihood value, however, the original function did not produce big parameters in absolute value that is the behavior expected when the “monotone likelihood” is verified. When we do not have large estimates, this indicates that we do not need use the penalized function.

Figure 1 shows the curves of the expected number of failures provided by the considered distributions in this work, including the Weibull model, in comparison with the original data. We can see, by visual analysis that the original *q*-Exponential provided the closest curve of real data and the Weibull distribution provided the worst results, with the furthest curve from real data. The corrected functions had similar performances.

In order to provide another comparison between the *q*-exponential (original and corrected ones) and the Weibull distributions, we computed the mean absolute error (MAE) for all the functions, the results for this first example are presented in Table 3. From Table 3, we see that the original *q*-Exponential present the best fit for the considered data (smallest MAE), the Weibull model had the worst performance (biggest MAE).

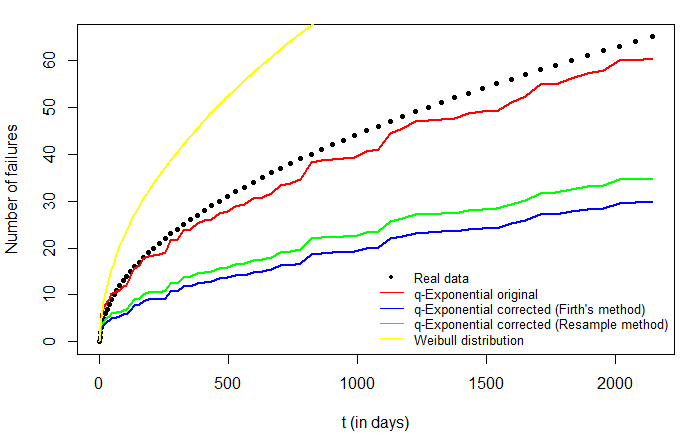


Figure 1 -Expected number of failures of original and corrected *q*-Exponential models and Weibull distribution compared to real data – Example 1.

Table 3 - MAE for the original and corrected q-Exponential’s and Weibull expected number of failures compared to real data – Example 1.

|  |  |
| --- | --- |
| Distribution | MAE |
| Original *q*-Exponential | 3.47 |
| *q*-Exponential (Firth’s method) | 18.58 |
| *q*-Exponential (Resample method) | 15.59 |
| Weibull | 21.85 |

4.2. *Example Application 2*

The data presented in Table 4 is the times between failure of an unspecified equipment.

Differently of the first application example, in Table 5 we can see that for these data the original *q*-Exponential log-likelihood presented very big estimates, which indicates the existence of the monotone likelihood for this dataset. The resample method also provided big estimates, this mean that this correction are not efficient to penalize the monotone likelihood of the original*q*-Exponential. On other hand, the function corrected by the Firth method presented well-behaved estimates.

Figure 2 present the curves of the expected number of failures for this example. Once more, the Weibull model had the worst result, and the function corrected by the Firth method had the best performance in this case, with the closest curve from real data.

Table 6 shows that the *q*-Exponential corrected by the Firth method presented the best performance for these data. The Weibull model present the worst result, with the biggest MAE. Thus, in this situation, the *q*-Exponential corrected by the Firths method is indicated.

Table 4 - TBFs of an unspecified equipment (in hours)

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **1** | 370 | **22** | 1085 | **43** | 1293 | **64** | 1522 | **85** | 1792 |
| **2** | 706 | **23** | 1102 | **44** | 1300 | **65** | 1522 | **86** | 1820 |
| **3** | 716 | **24** | 1102 | **45** | 1310 | **66** | 1530 | **87** | 1868 |
| **4** | 746 | **25** | 1108 | **46** | 1313 | **67** | 1540 | **88** | 1881 |
| **5** | 785 | **26** | 1115 | **47** | 1318 | **68** | 1560 | **89** | 1890 |
| **6** | 797 | **27** | 1120 | **48** | 1330 | **69** | 1567 | **90** | 1893 |
| **7** | 844 | **28** | 1134 | **49** | 1355 | **70** | 1578 | **91** | 1895 |
| **8** | 855 | **29** | 1140 | **50** | 1390 | **71** | 1594 | **92** | 1910 |
| **9** | 858 | **30** | 1199 | **51** | 1416 | **72** | 1602 | **93** | 1923 |
| **10** | 886 | **31** | 1200 | **52** | 1419 | **73** | 1604 | **94** | 1940 |
| **11** | 930 | **32** | 1200 | **53** | 1420 | **74** | 1608 | **95** | 1945 |
| **12** | 960 | **33** | 1203 | **54** | 1420 | **75** | 1630 | **96** | 2023 |
| **13** | 988 | **34** | 1222 | **55** | 1450 | **76** | 1642 | **97** | 2100 |
| **14** | 990 | **35** | 1235 | **56** | 1452 | **77** | 1674 | **98** | 2130 |
| **15** | 1000 | **36** | 1238 | **57** | 1475 | **78** | 1730 | **99** | 2215 |
| **16** | 1010 | **37** | 1252 | **58** | 1478 | **79** | 1750 | **100** | 2268 |
| **17** | 1010 | **38** | 1258 | **59** | 1481 | **80** | 1750 | **101** | 2240 |
| **18** | 1016 | **39** | 1262 | **60** | 1485 | **81** | 1763 |  |  |
| **19** | 1018 | **40** | 1269 | **61** | 1502 | **82** | 1768 |  |  |
| **20** | 1020 | **41** | 1270 | **62** | 1505 | **83** | 1781 |  |  |
| **21** | 1055 | **42** | 1290 | **63** | 1513 | **84** | 1782 |  |  |

Table 5 -Parameter estimates and log-likelihood values – Example 2

|  |  |  |  |
| --- | --- | --- | --- |
| Method |  |  | Log-likelihood value |
| Original | -2937219 | 6661615041 | -780.3920 |
| Firth’s | -14.4093 | 34948.4141 | -762.7225 |
| Resample | -6329638 | 14355622391 | -780.3920 |

Table 6 -MAE for the original and corrected *q*-Exponential’s and Weibull expected number of failures compared to real data – Example 2

|  |  |
| --- | --- |
| Distribution | MAE |
| Original *q*-Exponential | 15.88 |
| *q*-Exponential (Firth’s method) | 13.95 |
| *q*-Exponential (Resample method) | 15.95 |
| Weibull | 33.44 |

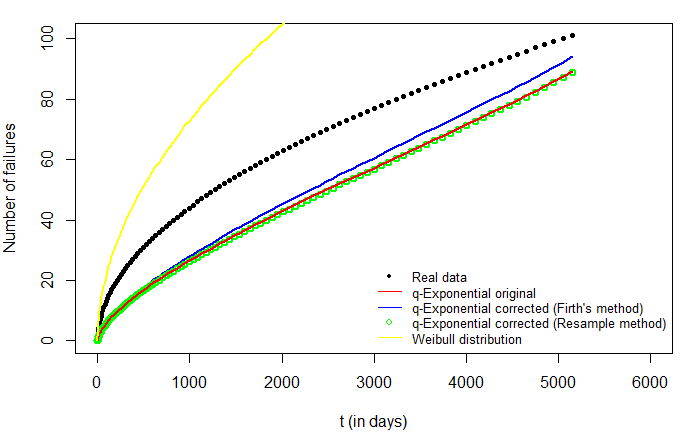


Figure 2 -Expected number of failures of original and corrected *q*-Exponential models and Weibull distribution compared to real data – Example 2.

1. **DISCUSSION**

We showed two real failure data and applied four distributions to fit these datasets. In both application examples the *q*-Exponential model was better, one was better modelled by the original *q*-Exponential, and the other was better modelled by the *q*-Exponential corrected by the Firth method.

The idea of this work is not to show that the *q*-Exponential (the original or the corrected ones) is better than the Weibull model, but to present different situations that must be analyzed through different perspectives. It means that, there are cases in which the original *q*-Exponential must be used, cases that the corrected *q*-Exponential is indicates and cases that the Weibull must be applied. Thus, the *q*-Exponential (original and corrected one) is an alternative for the Weibull in some situations.

The p-value, considering the two examples of the Section 4, for all distributions were greater than 0.20. This mean that all the functions used in this paper can model these datasets. Another important information is that all these datasets is referred to systems in the wear-out phase.

In reliability, to obtain a good fit for the data is very important, because information as number expected of failures in the future is computed based on the distributions. Hence, maintenance contracts are firmed based on this kind of forecast. Thus, the accuracy of the prevision will help the company to not spend unnecessary resources.

1. **CONCLUSION**

It was applied four functions to model two different reliability datasets in the wear-out phase. It was showed in [1] that, when the hazard rate is crescent, the *q*-Exponential may present a problem called “monotone likelihood”. Thus, we applied to model these datasets the following distributions: original *q*-exponential, *q*-Exponential corrected by the Firth method, *q*-Exponential corrected by the resample method and the Weibull distribution.

For the considered data in this work, the *q*-Exponential presented the monotone likelihood problem only once (second application example). For this dataset, which is considered a big sample (), the estimates provided by the original *q*-Exponential model is too big in absolute value, as can be seen in Table 5.

In order to make comparison between the distributions, we compute the mean absolute value (MAE). The original *q*-Exponential was the best in one situation and the function corrected by the Firth method was the best in another case. The *q*-Exponential corrected by the resample method was not the best in none of the analyzed situations.

The results achieved in this paper shows that there is not a distribution better in all situations. In practice, we must compare the performances and choose the model than can provide the more accurate information about the system interested. The *q*-Exponential model shows that it can be a good alternative for the Weibull distribution in some situations.

For future works, we intend apply the distributions used in this work in more reliability datasets in wear-out phase and analyze the results.

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1. **REFERENCES**

[1] NEGREIROS, A. C. S. V., LINS, I. D., SALES FILHO, R. L. M., MOURA, M. C., DROGUETT. E. L. Correction on the Log-Likelihood of the *q*-Exponential Distribution for use in the Reliability Context. ABRISCO, (2017).

[2]MALACARNE, L. C.; MENDES, R. S. & LENZI, E. K. *q*-exponential distribution in urban agglomeration. Physical Review E - Statistical, Nonlinear, and Soft Matter Physics, v. 65, n. 1, p. 1–3, (2002).

[3] CAMPO, M. G.; FERRI, G. L.; ROSTON, G. B. *q*-Exponential Distribution in Time Correlation Function of Water Hydrogen Bonds. Brazilian Journal of Physics, v. 39, n. 2, (2009).

[4] PATRIOTA, A. G. A *q*-Exponential regression model. Sankhyã: The Indian Journal of Statistics 2012, Volume 74-B, Part 1, pp. 149-170, (2012).

[5] LUDESCHER, J.; BUNDE A. Universal behavior of the interoccurrence times between losses in ﬁnancial markets: Independence of the time resolution. American Physical Society. Physical Review E, 90, 062809, (2014).

[6] SALES FILHO, R.; LOPES DROGUETT, E.; LINS, I.; MOURA, M. C.; AZEVEDO, R. Stress-strength reliability estimation based on the *q*-Exponential distribution. Quality and Reliability Engineering International, v. 4, p. 51, (2016).

[7] LINS, D. L.; MOURA, M. C.; DROGUETT, E. L.; CORRÊA, T. L. Combining Generalized Renewal Processes with Non-Extensive Entropy-Based q-Distributions for Reliability Applications. Entropy, 20, 223, (2018).

[8] MODARRES, M.; KAMINSKIY, M. & KRIVTSOV, V. Reliability Engineering and Risk Analisys. CRC Press, Ed. 3, 522p, (2016).

[9] ROSS, S. M. Simulation, 5th ed. San Diego: Academic Press, (2012).

[10] FIRTH, D. Bias Reduction of Maximum Likelihood Estimates. Biometrika Trust, v. 80, n. 1, p. 27–38, 1993.

[11] CRIBARI-NETO, F., FRERY, A. C., SILVA, M. F. Improved estimation of clutter properties in speckled imagery. Computational Statistics & Data Analisis, 40, 801-824. 2002.

[12] JEFFREYS, H. An invariant form for the prior probability in estimation problems. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, p. 453–461, (1946).

[13] PEREIRA, D. M. Determining the optimal replacement of complex medical equipment by means of obsolescence analysis (in Portuguese), 77 p. Master Thesis – Universidade Federal de Pernambuco/UFPE, Brazil, (2017).